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Chester B. Sensenig

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PLATE EQUATIONS AND FORMULAS FOR STRESSES, STRAINS, AND DISPLACEMENTS

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Results obtained at the Courant Institute of Mathematical Sciences, New York University, with the National Science Foundation, Grant No. NSF-GP-34620.

Introduction

In this report we derive estimates for the displacements, strains, stresses, and their derivatives in the interior of a thin plate. The estimates are given in terms of the maximum strains and the thickness and diameter of the plate. The estimates are used to derive formulas for the displacements, strains, and stresses in terms of the vertical displacement, w, of the middle surface and a stress function ψ. Also differential equations are obtained for w and ψ. To lowest order terms, these are the von Karman-Föppl equations. To higher order terms, they are accurate to the same order as those obtained by F. John [1] but are expressed in terms of different dependent variables. Several simplifications of the differential equations and formulas for the displacements, strains, and stresses are possible, one of which is presented. No accuracy is lost in this simplification which is valid for very small strains with relatively large bending permitted. Perhaps the most important of the results are the simplified plate equations (0.22').

The formulas derived for the displacements, stresses, and strains show several terms explicitly and the order of the remainder. To show all terms explicitly to a higher order would require the introduction of more than the five elastic parameters which we use here, and this was the criterion used to decide how many terms should be shown explicitly.

The assumptions made to obtain these results are that sufficiently many continuous derivatives of the displacements exist, the strain energy density function is an analytic function of the strain invariants, and the plate thickness-diameter ratio and the strains are sufficiently small.

Let u_1, u_2, u_3 be the displacements parallel to the coordinate axes (Cartesian) of the particle which was at (x_1, x_2, x_3) in the undeformed plate, and let

$$(0.1) \qquad e_{ij}(x_1, x_2, x_3) = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

be the strains where, $_{,i}$ denotes $\partial/\partial x_i$, Latin indices assume the values 1,2,3, and repeated indices are summed over their range. There will be some exceptions to the last two conventions which will be made clear by the context.

We let x_3 measure distance along the vertical axis and consider a plate which in the undeformed state is bounded by the planes $x_3 = \pm h$ and by a vertical surface at the edge.

The theory used is the exact three-dimensional theory for a homogeneous isotropic material having a strain energy density function which is analytic in the strain invariants. We introduce stresses t_{ij} (not the usual ones) by letting

$$(0.2) \qquad t_{ij} = \frac{\partial W}{\partial e_{ij}}$$

where W is the strain energy density function (energy per unit undeformed volume). If $\bar{S}$ is a surface in the deformed plate and S is the surface in the undeformed plate consisting of the same particles, then $t_{ij}\, n_j\, \vec{g}_i$ is the stress vector (force per unit undeformed area) acting across $\bar{S}$ where (n_1,n_2,n_3) is the unit normal vector to S pointing to the particles exerting the force across $\bar{S}$, and $\vec{g}_i = \frac{\partial}{\partial x_i}(x_1+u_1,\ x_2+u_2,\ x_3+u_3)$ is a tangent vector to the curve consisting of the particles which were on a line parallel to the ith coordinate axis in the undeformed plate.

We restrict ourselves to deformations in which the stress vector is zero at the faces and the body force vector is zero throughout the plate. This gives (see [1],[2])

$$t_{i3} = 0 \quad \text{for} \quad x_3 = \pm h \tag{0.3}$$

$$t_{ij,j} + 2e_{ik}\, t_{kj,j} + 2e_{ik,j}\, t_{kj} = W_{,i}\ . \tag{0.4}$$

Each of the functions to be estimated can be expressed uniquely as the sum of an even and an odd function of x_3. The even and odd parts will be estimated separately. For example, $t_{ij} = t_{ij1} + t_{ij2}$ where $t_{ij\omega}$ $(\omega = 1,2)$ is chosen to be even in x_3 if ω plus the number of threes among i,j is even, and otherwise $t_{ij\omega}$ is chosen to be odd in x_3, i.e. $t_{\alpha\beta 2}$, $t_{\alpha 31}$, and t_{332} are even functions of x_3 (Greek indices always assume the vlaues 1,2), and $t_{\alpha\beta 1}$, $t_{\alpha 32}$, and t_{331} are odd functions of x_3. Similarly, we define $e_{ij\omega}$, $u_{i\omega}$.

Next we introduce the parameters in terms of which the estimates will be given. Let D be the horizontal diameter of the unstrained plate, and let $\varepsilon_\omega = \max |e_{ij\omega}|$ where the maximum is taken over the entire plate and over all values of i,j. Let θ be a parameter such that $0 < \theta < 1$ and consider plates for which $h/D \leq \theta$ and deformations for which $\varepsilon_\omega \leq h^2/D^2$. For all sufficiently small values of θ, the results which we obtain are valid at all particles for which the undeformed distance to the edge of the plate is at least θD. How small is sufficiently small depends only on the strain energy density W and not on the deformation or the dimensions of the plate. Thus we are limiting ourselves to plates which are sufficiently thin, to deformations with sufficiently small strains, and to particles sufficiently far from the edge.

The estimates for the stresses and their derivatives are listed in (0.5). An expression of the type $\mathcal{O}(A)$ represents a quantity whose absolute value is less than or equal $\kappa|A|$ where κ is a positive dimensionless constant whose value depends only on θ and the strain energy density W. The numbers k_1, k_2, k_3 are any non-negative integers which make the notation meaningful. Thus $k_1+k_2 = k$ on some lines and $k_1+k_2+k_3 = k$ on other lines. A statement such as $(i,j) = (\alpha,\beta)$ means that i and j take on the values 1, 2. We let $\bar{\varepsilon}_1 = \varepsilon_1$, $\bar{\varepsilon}_2 = \varepsilon_2 + \frac{D^2}{h^2} \varepsilon_1^2$ and μ is one of the Lamé constants and is defined later in (0.9). The first two lines give estimates for all stresses and their derivatives, and the remaining lines give sharper estimates for some stresses and their derivatives.

$$
\left\{
\begin{aligned}
&\frac{\partial^k t_{ij1}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left(\dfrac{\mu\varepsilon_1}{D^k}\right) & \begin{cases} \text{if } k_3=0 \\ \text{if } k_3=1 \text{ and } i=3 \text{ or } j=3 \\ \text{if } k_3 \text{ is even and } (i,j)=(\alpha,\beta) \text{ or } (3,3) \\ \text{if } k_3 \text{ is odd and } (i,j) = (\alpha,3) \end{cases} \\ \mathcal{O}\left(\dfrac{\mu\varepsilon_1}{hD^{k-1}}\right) & \text{otherwise} \end{cases} \\
&\frac{\partial^k t_{ij2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left(\dfrac{\mu\bar{\varepsilon}_2}{D^k}\right) & \begin{cases} \text{if } k_3 = 0 \\ \text{if } k_3 = 1 \text{ and } i = 3 \text{ or } j = 3 \\ \text{if } k_3 \text{ is even and } (i,j)=(\alpha,\beta) \text{ or } (3,3) \\ \text{if } k_3 \text{ is odd and } (i,j)=(\alpha,3) \end{cases} \\ \mathcal{O}\left(\dfrac{\mu h\bar{\varepsilon}_2}{D^{k+1}}\right) & \text{otherwise} \end{cases} \\
&\frac{\partial^k t_{\alpha 31}}{\partial x_1^{k_1}\ \partial x_2^{k_2}} = \mathcal{O}\left(\frac{\mu h\varepsilon_1}{D^{k+1}}\right) \\
&\frac{\partial^k t_{\alpha 32}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{\mu h^{3-k_3}\bar{\varepsilon}_2}{D^{k+3-k_3}}\right) \quad \text{for } k_3 = 0,1,2 \\
&\frac{\partial^k t_{33\omega}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\frac{\mu h^{2-k_3}\bar{\varepsilon}_\omega}{D^{k+2-k_3}}\right) \quad \text{for } k_3 = 0,1
\end{aligned}
\right. \tag{0.5}
$$

for $k = 0,1,2,\ldots$.

The estimates for the strains and their derivatives are given in (0.6)

$$(0.6)\begin{cases} \dfrac{\partial^k e_{ij1}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left(\dfrac{\varepsilon_1}{D^k}\right) & \begin{cases} \text{if } k_3 = 0 \\ \text{if } k_3 \text{ is even and } (i,j)=(\alpha,\beta) \text{ or } (3,3 \\ \text{if } k_3 \text{ is odd and } (i,j) = (\alpha,3) \end{cases} \\ \mathcal{O}\left(\dfrac{\varepsilon_1}{hD^{k-1}}\right) & \text{otherwise} \end{cases} \\ \\ \dfrac{\partial^k e_{ij2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left(\dfrac{\bar{\varepsilon}_2}{D^k}\right) & \begin{cases} \text{if } k_3 = 0 \\ \text{if } k_3 \text{ is even and } (i,j)=(\alpha,\beta) \text{ or } (3,3 \\ \text{if } k_3 \text{ is odd and } (i,j) = (\alpha,3) \end{cases} \\ \mathcal{O}\left(\dfrac{h\,\bar{\varepsilon}_2}{D^{k+1}}\right) & \text{otherwise} \end{cases} \\ \\ \dfrac{\partial^k e_{\alpha 31}}{\partial x_1^{k_1}\;\partial x_2^{k_2}} = \mathcal{O}\left(\dfrac{h\,\varepsilon_1}{D^{k+1}}\right) \\ \\ \dfrac{\partial^k e_{\alpha 32}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \mathcal{O}\left(\dfrac{h^{3-k_3}\,\bar{\varepsilon}_2}{D^{k+3-k_3}}\right) \quad \text{for } k_3 = 0,1,2 \end{cases}$$

for $k = 0,1,2,\ldots$.

The estimates for the displacements and their derivatives are given in (0.7). For (0.7) to be valid we must first perform a rigid transformation so that $u_i = 0$, $u_{\alpha,3} = 0$, and $u_{2,1} = 0$ all at some particle whose undeformed distance from the edge is at least θD.

$$(0.7)\begin{cases} \dfrac{\partial^k u_{i1}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \left(\dfrac{\varepsilon_1}{D^{k-1}}\right)\begin{cases}\text{if } k_3 \text{ is even and } i = \alpha \\ \text{if } k_3 \text{ is odd and } i = 3\end{cases} \\ \left(\dfrac{\varepsilon_1}{hD^{k-2}}\right) \quad \text{otherwise} \end{cases} \\ \\ \dfrac{\partial^k u_{i2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \left(\dfrac{\bar{\varepsilon}_2}{D^{k-1}}\right)\begin{cases}\text{if } k_3 \text{ is even and } i = \alpha \\ \text{if } k_3 \text{ is odd and } i = 3\end{cases} \\ \left(\dfrac{h\,\bar{\varepsilon}_2}{D^{k}}\right) \quad \text{otherwise} \end{cases} \end{cases}$$

for $k = 0,1,2,\ldots$.

To define the elastic parameters we will use, we first introduce the strain invariants by

$$(0.8) \qquad s_1 = e_{ii}\ , \quad s_2 = e_{ij}\, e_{ji}\ , \quad s_3 = e_{ij}\, e_{jk}\, e_{ki}\ .$$

Then the elastic parameters are

$$(0.9) \qquad \begin{cases} \lambda = \left.\dfrac{\partial^2 W}{\partial s_1^2}\right|_{e_{ij}=0} \\ \mu = \left.\dfrac{\partial W}{\partial s_2}\right|_{e_{ij}=0} \\ A = \left.\dfrac{1}{2}\,\dfrac{\partial^3 W}{\partial s_1^3}\right|_{e_{ij}=0} \\ B = \left.\dfrac{\partial^2 W}{\partial s_1\ \partial s_2}\right|_{e_{ij}=0} \\ C = \left.3\,\dfrac{\partial W}{\partial s_3}\right|_{e_{ij}=0} \end{cases}$$

To define the stress function we first define $T_{\alpha\beta}(x_1,x_2)$ by

$$(0.10) \qquad T_{\alpha\beta} = \frac{1}{4\mu h} \int_{-h}^{h} (t_{\alpha\beta} + 2e_{\alpha k}t_{k\beta} - W\delta_{\alpha\beta})\, dx_3 \; .$$

Later we will observe that $T_{\alpha\beta} = T_{\beta\alpha}$ and $T_{\alpha\beta,\beta} = 0$. Hence there is a stress function $\psi(x_1,x_2)$ such that

$$(0.11) \qquad T_{11} = \psi_{,22} \; ; \quad T_{22} = \psi_{,11} \; ; \quad T_{12} = -\psi_{,12} \; .$$

Let

$$(0.12) \qquad \begin{cases} v_\alpha(x_1,x_2) = u_\alpha\big|_{x_3=0} \\ w\;(x_1,x_2) = u_3\big|_{x_3=0} \end{cases}$$

be the displacements of the middle surface.

Many combinations of the elastic constants (0.9) occur and are denoted by C_k $(k = 1,2,\ldots,190)$. The definitions of the constants can be found in section 4.

The horizontal displacements v_α of the middle surface are determined by (0.13). By integrating the first result, the v_α are determined to within $\mathcal{O}(\frac{h^2}{D}\bar{\varepsilon}_2)$. Using this result to eliminate v_α on the right-hand side of the second result, v_α can then be determined to within order $\mathcal{O}(\frac{h^4}{D^3}\bar{\varepsilon}_2)$. Thus v_α is determined by an iteration process in terms of w and ψ.

In (0.13) and the following the operator Δ is defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \; .$$

$$
(0.13)\quad \begin{cases}
v_{\alpha,\beta\gamma} = -\, w_{,\alpha} w_{,\beta\gamma} + T_{\alpha\beta,\gamma} + T_{\alpha\gamma,\beta} - T_{\gamma\beta,\alpha} \\
\qquad - \dfrac{\lambda}{3\lambda+2\mu} \left(\delta_{\alpha\beta}\Delta\psi_{,\gamma} + \delta_{\alpha\gamma}\Delta\psi_{,\beta} - \delta_{\gamma\beta}\Delta\psi_{,\alpha}\right) + \mathcal{O}\left(\dfrac{h^2}{D^3}\,\bar{\varepsilon}_2\right) \\
v_{\alpha,\beta\gamma} = -\, w_{,\alpha} w_{,\beta\gamma} - v_{\omega,\alpha} v_{\omega,\beta\gamma} + S_{\alpha\beta,\gamma} + S_{\alpha\gamma,\beta} - S_{\gamma\beta,\alpha} + \mathcal{O}\left(\dfrac{h^4}{D^5}\,\bar{\varepsilon}_2\right) \\
S_{\alpha\beta} = \delta_{\alpha\beta} \Big\{ \dfrac{-\lambda}{3\lambda+2\mu}\,\Delta\psi + C_{29}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{30}(\Delta\psi)^2 \\
\qquad + h^2\left[C_{31} w_{,\gamma\nu} w_{,\gamma\nu} + C_{32}(\Delta w)^2\right] \Big\} + T_{\alpha\beta} - C_{21} T_{\alpha\beta}\Delta\psi - C_{22} T_{\alpha\gamma} T_{\gamma\beta} \\
\qquad - h^2\left[\dfrac{\lambda}{6(3\lambda+2\mu)}\,\Delta\psi_{,\alpha\beta} + C_{23} w_{,\alpha\beta}\Delta w + C_{24} w_{,\alpha\gamma} w_{,\gamma\beta}\right]
\end{cases}
$$

The even and odd parts of the displacements are given by (0.14) and (0.15)

$$
\left\{
\begin{aligned}
u_{\alpha 1} = {} & -x_3 w_{,\alpha} \Bigl\{ 1 - \frac{\lambda}{3\lambda+2\mu} \Delta\psi + C_{39} \psi_{,\gamma\nu} \psi_{,\gamma\nu} + C_{43} (\Delta\psi)^2 + T_{\gamma\nu} w_{,\gamma} w_{,\nu} \\
& - w_{,\gamma} w_{,\gamma} \Bigl[\frac{1}{2} + \frac{\lambda}{2(3\lambda+2\mu)} \Delta\psi + \frac{1}{8} w_{,\nu} w_{,\nu} \Bigr] + (C_{45} x_3^2 + C_{46} h^2) w_{,\gamma} \Delta w_{,\gamma} \\
& - (C_{95} x_3^2 + C_{96} h^2) w_{,\gamma\nu} w_{,\gamma\nu} - (C_{97} x_3^2 + C_{98} h^2) (\Delta w)^2 \Bigr\} \\
& + \left[\frac{3\lambda+4\mu}{6(\lambda+2\mu)} x_3^3 - \frac{2(\lambda+\mu)}{\lambda+2\mu} h^2 x_3 \right] \bigl[\Delta w_{,\alpha} (1 + \frac{1}{2} w_{,\gamma} w_{,\gamma}) - v_{\gamma,\alpha} \Delta w_{,\gamma} \bigr] \\
& + (C_{88} x_3^3 - \frac{\lambda}{\lambda+2\mu} x_3 h^2) w_{,\gamma} \Delta\psi_{,\gamma\alpha} + (C_{89} x_3^3 + \frac{C_{75}}{2\mu} x_3 h^2) w_{,\alpha\gamma} \Delta\psi_{,\gamma} \\
& + \Bigl[C_{90} \frac{x_3^5}{h^2} + C_{91} x_3^3 + \frac{C_{82}}{2\mu} x_3 h^2 \Bigr] T_{\gamma\nu,\alpha} w_{,\gamma\nu} \\
& + (C_{90} \frac{x_3^5}{h^2} + C_{92} x_3^3 + \frac{C_{83}}{2\mu} x_3 h^2) T_{\gamma\nu} w_{,\gamma\nu\alpha} \\
& + (C_{93} x_3^3 + \frac{C_{79}}{2\mu} x_3 h^2) \Delta w \, \Delta\psi_{,\alpha} + (C_{94} x_3^3 - C_{86} x_3 h^2) \Delta\psi \, \Delta w_{,\alpha} \\
& + (\frac{x_3^3}{3} - x_3 h^2) \bigl[-w_{,\gamma} \Delta T_{\alpha\gamma} + w_{,\gamma} w_{,\gamma\nu} w_{,\alpha\nu} + C_{87} T_{\alpha\gamma} \Delta w_{,\gamma} \\
& + (2 + \frac{C}{2\mu}) w_{,\gamma\nu} T_{\gamma\alpha,\nu} \bigr] + \frac{\lambda}{6(\lambda+2\mu)} (x_3^3 - 6 x_3 h^2) w_{,\gamma\alpha} w_{,\gamma} \Delta w \\
& - x_3 v_{\gamma,\alpha} \Bigl\{ -w_{,\gamma} (1 - \frac{\lambda}{3\lambda+2\mu} \Delta\psi - \frac{1}{2} w_{,\nu} w_{,\nu}) + w_{,\nu} v_{\nu,\gamma} \Bigr\} + \mathcal{O}(\frac{h^6}{D^5} \varepsilon_1) \\
\\
u_{\alpha 2} = {} & v_\alpha + x_3^2 \Bigl\{ \frac{\lambda}{2(3\lambda+2\mu)} \Delta\psi_{,\alpha} - C_{39} \psi_{,\gamma\nu} \psi_{,\gamma\nu\alpha} + C_{173} \Delta\psi \, \Delta\psi_{,\alpha} \\
& - \frac{\lambda}{2(\lambda+2\mu)} w_{,\alpha} \Delta w - \frac{1}{2} C_4 w_{,\alpha} \Delta\psi_{,\gamma} w_{,\gamma} \\
& - \frac{1}{2} C_5 w_{,\alpha} \Delta\psi \, \Delta w - v_{\gamma,\alpha} \left[\frac{\lambda}{2(3\lambda+2\mu)} \Delta\psi_{,\gamma} - \frac{\lambda}{2(\lambda+2\mu)} w_{,\gamma} \Delta w \right] \\
& - (\frac{1}{2} C_1 + \frac{1}{4} C_2 \frac{x_3^2}{h^2}) w_{,\alpha} T_{\gamma\nu} w_{,\gamma\nu} + (C_{174} x_3^2 + C_{175} h^2) w_{,\gamma\nu} w_{,\gamma\nu\alpha} \\
& + (C_{176} x_3^2 + C_{177} h^2) \Delta w \, \Delta w_{,\alpha} + (C_{178} x_3^2 + C_{179} h^2) w_{,\gamma\alpha} \Delta w_{,\gamma} \Bigr\} \\
& + \mathcal{O}(\frac{h^6}{D^5} \bar{\varepsilon}_2)
\end{aligned}
\right.
\tag{0.14}
$$

$$
\left\{
\begin{aligned}
u_{31} &= w + \frac{x_3^2}{2}\left\{\frac{\lambda}{\lambda+2\mu}\,\Delta w + \left(C_1 + \frac{1}{2}\,C_2\,\frac{x_3^2}{h^2}\right)T_{\gamma\nu}w_{,\gamma\nu}\right.\\
&\qquad \left. + C_4\Delta\psi_{,\gamma}w_{,\gamma} + C_5\Delta\psi\Delta w\right\} + \mathcal{O}\left(\frac{h^5}{D^4}\,\varepsilon_1\right)\\
u_{32} &= x_3\left\{-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi + C_{39}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{43}(\Delta\psi)^2 + T_{\gamma\nu}w_{,\gamma}w_{,\nu}\right.\\
&\qquad - w_{,\gamma}w_{,\gamma}\left[\frac{1}{2} + \frac{\lambda}{2(3\lambda+2\mu)}\,\Delta\psi + \frac{1}{8}\,w_{,\nu}w_{,\nu}\right]\\
&\quad + \left(\frac{1}{3}\,C_7x_3^2 + C_{41}h^2\right)w_{,\gamma\nu}w_{,\gamma\nu} + \left(C_{44}x_3^2 + C_{42}h^2\right)(\Delta w)^2\\
&\quad \left. + \left(C_{45}x_3^2 + C_{46}h^2\right)w_{,\gamma}\Delta w_{,\gamma}\right\} + \mathcal{O}\left(\frac{h^5}{D^4}\,\bar{\varepsilon}_2\right) \quad .
\end{aligned}
\right.
\tag{0.15}
$$

The strains are given by (0.16), (0.17), (0.18).

$$
\left\{
\begin{aligned}
e_{\alpha\beta 1} &= x_3\left\{-w_{,\alpha\beta} + \left[\frac{3\lambda+4\mu}{6(\lambda+2\mu)}\,x_3^2 - \frac{2(\lambda+\mu)}{\lambda+2\mu}\,h^2\right]\Delta w_{,\alpha\beta}\right.\\
&\quad + w_{,\alpha\beta}\left(\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi - \frac{1}{2}\,w_{,\gamma}w_{,\gamma}\right) + w_{,\gamma}\left(T_{\gamma\beta,\alpha} + T_{\alpha\gamma,\beta} - T_{\alpha\beta,\gamma}\right)\\
&\quad \left. - \frac{\lambda}{3\lambda+2\mu}\left(w_{,\beta}\Delta\psi_{,\alpha} + w_{,\alpha}\Delta\psi_{,\beta} - \delta_{\alpha\beta}w_{,\gamma}\Delta\psi_{,\gamma}\right)\right\} + \mathcal{O}\left(\frac{h^4}{D^4}\,\varepsilon_1\right)\\
e_{\alpha\beta 2} &= \delta_{\alpha\beta}\left\{-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi + C_{29}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{30}(\Delta\psi)^2\right.\\
&\quad \left. + h^2\left[C_{31}w_{,\gamma\nu}w_{,\gamma\nu} + C_{32}(\Delta w)^2\right]\right\}\\
&\quad + T_{\alpha\beta} - C_{21}T_{\alpha\beta}\Delta\psi - C_{22}T_{\alpha\gamma}T_{\gamma\beta} + \frac{\lambda}{2(3\lambda+2\mu)}\left(x_3^2 - \frac{h^2}{3}\right)\Delta\psi_{,\alpha\beta}\\
&\quad - \left[\frac{\lambda}{2(\lambda+2\mu)}\,x_3^2 + C_{23}h^2\right]w_{,\alpha\beta}\Delta w + \left(\frac{1}{2}\,x_3^2 - C_{24}h^2\right)w_{,\alpha\gamma}w_{,\gamma\beta}\\
&\quad + \mathcal{O}\left(\frac{h^4}{D^4}\,\bar{\varepsilon}_2\right).
\end{aligned}
\right.
\tag{0.16}
$$

$$(0.17)\begin{cases} e_{\alpha 31} = \frac{1}{2}\,(x_3^2-h^2)\,\Big\{\frac{2(\lambda+\mu)}{\lambda+2\mu}\,(1+\frac{1}{2}\,w_{,\gamma}w_{,\gamma})\Delta w_{,\alpha} \\ \quad + \frac{\lambda}{\lambda+2\mu}\,(w_{,\gamma}w_{,\gamma\alpha}\Delta w + w_{,\gamma}\Delta\psi_{,\gamma\alpha}) \\ \quad - \frac{C_{75}}{2\mu}\,w_{,\alpha\gamma}\Delta\psi_{,\gamma} - (\frac{C_{82}}{2\mu} + \frac{1}{2}\,\frac{x_3^2}{h^2})\,T_{\gamma\nu,\alpha}w_{,\gamma\nu} \\ \quad - (\frac{C_{83}}{2\mu} + \frac{1}{2}\,\frac{x_3^2}{h^2})\,T_{\gamma\nu}w_{,\gamma\nu\alpha} - \frac{C_{79}}{2\mu}\,\Delta w\Delta\psi_{,\alpha} + C_{86}\Delta\psi\Delta w_{,\alpha} - w_{,\gamma}\Delta T_{\alpha\gamma} \\ \quad + w_{,\gamma}w_{,\gamma\nu}w_{'\alpha\nu} + C_{87}T_{\alpha\gamma}\,\Delta w_{,\gamma} + (2+\frac{C}{2\mu})\,w_{,\gamma\nu}T_{\gamma\alpha,\nu} \\ \quad + \frac{\lambda}{4(\lambda+\mu)}\,w_{,\alpha}[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2]\Big\} + \mathcal{O}(\frac{h^5}{D^5}\,\varepsilon_1) \\ e_{\alpha 32} = x_3(x_3^2-h^2)\,\Big\{\frac{C_{165}}{2\mu}\,w_{,\gamma\nu}w_{,\gamma\nu\alpha} + C_{171}\Delta w\Delta w_{,\alpha} + C_{172}w_{,\alpha\gamma}\Delta w_{,\gamma}\Big\} \\ \quad + \mathcal{O}(\frac{h^5}{D^5}\,\bar{\varepsilon}_2) \end{cases}$$

$$(0.18)\begin{cases} e_{331} = x_3\Big\{\frac{\lambda}{\lambda+2\mu}\,\Delta w + (C_1 + C_2\,\frac{x_3^2}{h^2})\,T_{\gamma\nu}w_{,\gamma\nu} \\ \quad + \frac{\lambda}{\lambda+2\mu}\,(w_{,\gamma}\Delta\psi_{,\gamma} + \frac{1}{2}\,w_{,\gamma}w_{,\gamma}\Delta w) + C_3\Delta\psi\Delta w\Big\} + \mathcal{O}(\frac{h^4}{D^4}\,\varepsilon_1) \\ e_{332} = -\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi + C_{39}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{40}(\Delta\psi)^2 + (C_7x_3^2+C_{41}h^2)\,w_{,\gamma\nu}w_{,\gamma\nu} \\ \quad + (C_8x_3^2+C_{42}h^2)\,(\Delta w)^2 + \mathcal{O}(\frac{h^4}{D^4}\,\bar{\varepsilon}_2). \end{cases}$$

The stresses are given by (0.19), (0.20), (0.21).

$$
(0.19)\left\{\begin{aligned}
t_{\alpha\beta 1} = {} & \delta_{\alpha\beta} x_3 \Big\{ -\frac{2\lambda\mu}{\lambda+2\mu}\,\Delta w\left(1+\frac{1}{2}\,w_{,\gamma} w_{,\gamma}\right) - \frac{4\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\, w_{,\gamma}\Delta\psi_{,\gamma} \\
& + \left(C_{69} + \lambda C_{67}\,\frac{x_3^2}{h^2}\right) T_{\gamma\nu} w_{,\gamma\nu} + C_{70}\Delta\psi\,\Delta w \Big\} \\
& + x_3 \Big\{ -2\mu w_{,\alpha\beta} + \left[\frac{\mu(3\lambda+4\mu)}{3(\lambda+2\mu)}\,x_3^2 - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\,h^2\right]\Delta w_{,\alpha\beta} + C_{71} w_{,\alpha\beta}\Delta\psi \\
& - \mu w_{,\gamma} w_{,\gamma} w_{,\alpha\beta} + 2\mu w_{,\gamma}\left(T_{\gamma\beta,\alpha} + T_{\alpha\gamma,\beta} - T_{\alpha\beta,\gamma}\right) \\
& - \frac{2\lambda\mu}{3\lambda+2\mu}\left(w_{,\beta}\Delta\psi_{,\alpha} + w_{,\alpha}\Delta\psi_{,\beta}\right) - \frac{4\mu B}{\lambda+2\mu}\,T_{\alpha\beta}\,\Delta w \\
& - C\left(w_{,\alpha\gamma}T_{\gamma\beta} + w_{,\beta\gamma}T_{\gamma\alpha}\right)\Big\} + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\varepsilon_1\right) \\
t_{\alpha\beta 2} = {} & \delta_{\alpha\beta}\Big\{ C_{33}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{34}(\Delta\psi)^2 + \left(C_{12}x_3^2 + C_{35}h^2\right) w_{,\gamma\nu} w_{,\gamma\nu} \\
& + \left(C_{13}x_3^2 + C_{36}h^2\right)(\Delta w)^2\Big\} + 2\mu T_{\alpha\beta} + C_{37}T_{\alpha\beta}\Delta\psi \\
& + C_{38}T_{\alpha\gamma}T_{\gamma\beta} + \frac{\lambda\mu}{3\lambda+2\mu}\left(x_3^2 - \frac{h^2}{3}\right)\Delta\psi_{,\alpha\beta} \\
& + \left(C_{15}x_3^2 - 2\mu C_{23}h^2\right) w_{,\alpha\beta}\Delta w + \left(C_{16}x_3^2 - 2\mu C_{24}h^2\right) w_{,\alpha\gamma} w_{,\gamma\beta} \\
& + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\bar{\varepsilon}_2\right)
\end{aligned}\right.
$$

$$
(0.20)\left\{\begin{aligned}
t_{\alpha 3 1} = {} & \frac{1}{2}\left(x_3^2 - h^2\right)\Big\{ \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(1+\frac{1}{2}\,w_{,\gamma}w_{,\gamma}\right)\Delta w_{,\alpha} \\
& + \frac{2\lambda\mu}{\lambda+2\mu}\left(w_{,\gamma}w_{,\gamma\alpha}\Delta w + w_{,\gamma}\Delta\psi_{,\gamma\alpha}\right) - C_{75}w_{,\alpha\gamma}\Delta\psi_{,\gamma} \\
& - \left(C_{82} + \mu\,\frac{x_3^2}{h^2}\right) T_{\gamma\nu,\alpha} w_{,\gamma\nu} - \left(C_{83} + \mu\,\frac{x_3^2}{h^2}\right) T_{\gamma\nu} w_{,\gamma\nu\alpha} \\
& - C_{79}\Delta w\Delta\psi_{,\alpha} - C_{84}\Delta\psi\,\Delta w_{,\alpha} - 2\mu\left(w_{,\gamma}\Delta T_{\alpha\gamma} - w_{,\gamma}w_{,\gamma\nu}w_{,\alpha\nu}\right) \\
& + C_{85}T_{\alpha\gamma}\Delta w_{,\gamma} + (4\mu + C) w_{,\gamma\nu}T_{\gamma\alpha,\nu} \\
& + \frac{\lambda\ \mu}{2(\lambda+\mu)}\,w_{,\alpha}\left[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2\right]\Big\} + \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\varepsilon_1\right) \\
t_{\alpha 3 2} = {} & x_3\left(x_3^2 - h^2\right)\Big\{ C_{165}w_{,\gamma\nu}w_{,\gamma\nu\alpha} + C_{167}\Delta w\,\Delta w_{,\alpha} + C_{170}w_{,\alpha\gamma}\Delta w_{,\gamma}\Big\} \\
& \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\bar{\varepsilon}_2\right)
\end{aligned}\right.
$$

$$
(0.21)\left\{\begin{aligned}
t_{331} = {} & x_3(h^2-x_3^2)\Big\{\frac{\mu}{10}\Big(1-\frac{x_3^2}{h^2}\Big)(\Delta T_{\gamma\nu}w_{,\gamma\nu} + 2T_{\gamma\nu,\omega}w_{,\gamma\nu\omega}) \\
& + C_{148}w_{,\gamma\nu}w_{,\gamma\nu}\Delta w + C_{149}\Delta\psi_{,\gamma\nu}w_{,\gamma\nu} + \frac{\mu}{2h^2}T_{\gamma\nu}w_{,\gamma\nu}w_{,\omega}w_{,\omega} \\
& + \Big(C_{150} - \frac{\mu}{10}\frac{x_3^2}{h^2}\Big)T_{\gamma\nu}\Delta w_{,\gamma\nu} + C_{151}(\Delta w)^3 + C_{152}\frac{1}{h^2}T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi \\
& + \frac{1}{3}(2\mu+C)w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma} \\
& + \frac{\mu}{h^2}\Big(T_{\gamma\nu}w_{,\gamma\nu} + \frac{1}{2}\psi_{,\gamma\nu}\psi_{,\gamma\nu}\Delta w - 2T_{\gamma\nu}T_{\nu\omega}w_{,\omega\gamma} \\
& \qquad\qquad - 2T_{\gamma\nu}T_{\omega\gamma,\nu}w_{,\omega} + \psi_{,\gamma\nu}\psi_{,\gamma\nu\omega}w_{,\omega}\Big) \\
& + \frac{\lambda\mu}{2(3\lambda+2\mu)}\frac{1}{h^2}\big[4T_{\gamma\nu}\Delta\psi_{,\gamma}w_{,\nu} - (\Delta\psi)^2\Delta w - 2\Delta\psi\Delta\psi_{,\gamma}w_{,\gamma}\big]\Big\} + \Big(\frac{\mu h^6}{D^6}\varepsilon_1\Big) \\
t_{332} = {} & (x_3^2-h^2)\Big\{\mu w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda\mu}{\lambda+2\mu}(\Delta w)^2 + C_{182}\Delta\psi(\Delta w)^2 \\
& + C_{183}\Delta\psi w_{,\gamma\nu}w_{,\gamma\nu} + \Big(C_{189} + C_{185}\frac{x_3^2}{h^2}\Big)T_{\gamma\nu}w_{,\gamma\nu}\Delta w \\
& + (\mu+C)w_{,\gamma\nu}w_{,\nu\omega}T_{\omega\gamma} + \frac{4\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)}w_{,\gamma}\Delta\psi_{,\gamma}\Delta w \\
& \qquad\qquad + \frac{\lambda\mu}{\lambda+2\mu}w_{,\gamma}w_{,\gamma}(\Delta w)^2 \\
& + (C_{186}x_3^2 + C_{190}h^2)w_{,\gamma\nu}\Delta w_{,\gamma\nu} + \mu w_{,\gamma}w_{,\gamma}w_{,\nu\omega}w_{,\nu\omega} \\
& + 2\mu w_{,\gamma}w_{,\nu\omega}(T_{\nu\omega,\gamma} - 2T_{\gamma\nu,\omega}) + \frac{4\lambda\mu}{3\lambda+2\mu}\Delta\psi_{,\gamma}w_{,\nu}w_{,\gamma\nu} \\
& \qquad\qquad + \frac{\lambda\mu}{3\lambda+2\mu}T_{\gamma\nu}\Delta\psi_{,\gamma\nu} \\
& + (x_3^2-h^2)\Big(C_{188}\Delta w_{,\gamma}\Delta w_{,\gamma} - \frac{1}{4}C_{165}w_{,\gamma\nu\omega}w_{,\gamma\nu\omega}\Big)\Big\} + \Big(\frac{\mu h^6}{D^6}\bar{\varepsilon}_2\Big)
\end{aligned}\right.
$$

The differential equations for w and ψ are listed in (0.22).

$$
\left\{
\begin{aligned}
\Delta^2\psi = {} & C_{57}\psi_{,\gamma\nu\omega}\psi_{,\gamma\nu\omega} + C_{58}\psi_{,\gamma\nu}\Delta\psi_{,\gamma\nu} + C_{59}\Delta\psi_{,\gamma}\Delta\psi_{,\gamma} \\
& + C_{60}T_{\gamma\nu}\Delta\psi_{,\gamma\nu} + \frac{3\lambda+2\mu}{4(\lambda+\mu)}[w_{,\gamma\nu}w_{,\gamma\nu}-(\Delta w)^2][1+w_{,\omega}w_{,\omega}+C_{61}\Delta\psi] \\
& - \frac{3\lambda+2\mu}{2(\lambda+\mu)} w_{,\gamma}w_{,\omega\nu}(2T_{\gamma\omega,\nu} - T_{\omega\nu,\gamma}) + \frac{\lambda}{\lambda+\mu} w_{,\gamma}w_{,\gamma\nu}\Delta\psi_{,\nu} \\
& - \frac{2\lambda+\mu}{\lambda+\mu} w_{,\gamma}\Delta w\Delta\psi_{,\gamma} + C_{62}T_{\gamma\nu}w_{,\gamma\nu}\Delta w \\
& + h^2(C_{63}w_{,\gamma\nu\omega}w_{,\gamma\nu\omega} + C_{64}w_{,\gamma\nu}\Delta w_{,\gamma\nu} + C_{65}\Delta w_{,\gamma}\Delta w_{,\gamma}) + \mathcal{O}(\frac{h^4}{D^6}\bar{\varepsilon}_2) \\[2ex]
\frac{2(\lambda+\mu)}{3(\lambda+2\mu)} h^2\Delta^2 w = {} & T_{\gamma\nu}w_{,\gamma\nu} + \frac{1}{2}\psi_{,\gamma\nu}\psi_{,\gamma\nu}\Delta w - 2T_{\gamma\nu}T_{\nu\omega}w_{,\omega\gamma} \\
& - 2T_{\gamma\nu}T_{\omega\gamma,\nu}w_{,\omega} + \psi_{,\gamma\nu}\psi_{,\gamma\nu\omega}w_{,\omega} \\
& + \frac{\lambda}{2(3\lambda+2\mu)}[4T_{\gamma\nu}\Delta\psi_{,\gamma}w_{,\nu} - (\Delta\psi)^2\Delta w - 2\Delta\psi\,\Delta\psi_{,\gamma}w_{,\gamma}] \\
& + C_{153}T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi \\
& + h^2[C_{154}w_{,\gamma}w_{,\gamma\nu}\Delta w_{,\nu} + C_{155}w_{,\gamma}\Delta w_{,\gamma}\Delta w + C_{156}w_{,\gamma}w_{,\nu\omega}w_{,\gamma\nu\omega} \\
& + C_{157}\Delta\psi_{,\gamma}\Delta w_{,\gamma} + C_{158}\Delta T_{\gamma\nu}w_{,\gamma\nu} + C_{159}T_{\gamma\nu,\omega}w_{,\gamma\nu\omega} \\
& + C_{160}w_{,\gamma\nu}w_{,\gamma\nu}\Delta w + C_{161}\Delta\psi_{,\gamma\nu}w_{,\gamma\nu} + C_{162}T_{\gamma\nu}\Delta w_{,\gamma\nu} \\
& + C_{163}(\Delta w)^3 - \frac{2}{3}w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma}] + \mathcal{O}(\frac{h^5}{D^6}\varepsilon_1)
\end{aligned}
\right.
\tag{0.22}
$$

The simplification of the results which we present here is valid for very small strains with relatively large bending permitted. More precisely, we consider the case $\varepsilon_1 = \mathcal{O}(\frac{h^3}{D^3})$ and $\varepsilon_2 = \mathcal{O}(\frac{h^4}{D^4})$. Then $\bar{\varepsilon}_2 = \mathcal{O}(\frac{h^4}{D^4})$. Since ε_1 is permitted to be larger than ε_2, we say that relatively large bending is permitted.

The simplified version of (0.13) no longer involves an iteration and is listed as (0.13').

$$(0.13')\left\{\begin{aligned}
v_{\alpha,\beta\gamma} = {} & -w_{,\alpha}w_{,\beta\gamma} + T_{\alpha\beta,\gamma} + T_{\alpha\gamma,\beta} - T_{\gamma\beta,\alpha} \\
& - \frac{\lambda}{3\lambda+2\mu}\,(\delta_{\alpha\beta}\Delta\psi_{,\gamma} + \delta_{\alpha\gamma}\,\Delta\psi_{,\beta} - \delta_{\alpha\beta}\,\Delta\psi_{,\alpha}) \\
& + 2h^2C_{31}w_{,\nu\omega}(\delta_{\alpha\beta}w_{,\nu\omega\gamma}+\delta_{\alpha\gamma}w_{,\nu\omega\beta}-\delta_{\gamma\beta}w_{,\nu\omega\alpha}) \\
& + 2h^2C_{32}\Delta w(\delta_{\alpha\beta}\Delta w_{,\gamma}+\delta_{\alpha\beta}\Delta w_{,\beta}-\delta_{\gamma\beta}\Delta w_{,\alpha}) \\
& - \frac{\lambda}{6(3\lambda+2\mu)}\,h^2\Delta\psi_{,\alpha\beta\gamma} \\
& - C_{23}h^2(w_{,\alpha\beta\gamma}\Delta w+w_{,\alpha\beta}\Delta w_{,\gamma}+ w_{,\alpha\gamma}\Delta w_{,\beta}- w_{,\gamma\beta}\Delta w_{,\alpha}) \\
& - C_{24}h^2(w_{,\alpha\nu\beta}w_{,\nu\gamma}+ 2w_{,\alpha\nu}w_{,\nu\beta\gamma}- w_{,\gamma\nu}w_{,\nu\beta\alpha}) + \mathcal{O}(\frac{h^4}{D^5}\,\bar{\varepsilon}_2)
\end{aligned}\right.$$

Similarly we list the remaining simplified results.

$$(0.14')\left\{\begin{aligned}
u_{\alpha 1} = {} & -x_3w_{,\alpha}\left\{1 - \frac{\lambda}{3\lambda+2\mu}\,\Delta\psi - \frac{1}{2}\,w_{,\gamma}w_{,\gamma}\right\} \\
& + \left[\frac{3\lambda+4\mu}{6(\lambda+2\mu)}\,x_3^3 - \frac{2(\lambda+\mu)}{\lambda+2\mu}\,x_3h^2\right]\Delta w_{,\alpha} + x_3v_{\gamma,\alpha}w_{,\gamma} + \mathcal{O}(\frac{h^6}{D^5}\,\varepsilon_1) \\
u_{\alpha 2} = {} & v_\alpha + x_3^2\left\{\frac{\lambda}{2(3\lambda+2\mu)}\,\Delta\psi_{,\alpha} - \frac{\lambda}{2(\lambda+2\mu)}\,w_{,\alpha}\,\Delta w\right. \\
& + (C_{174}x_3^2 + C_{175}h^2)w_{,\gamma\nu}w_{,\gamma\nu\alpha} + (C_{176}x_3^2 + C_{177}h^2)\Delta w\Delta w_{,\alpha} \\
& \left. + (C_{178}x_3^2 + C_{179}h^2)w_{,\gamma\alpha}\Delta w_{,\gamma}\right\} + \mathcal{O}(\frac{h^6}{D^5}\,\bar{\varepsilon}_2)
\end{aligned}\right.$$

$$(0.15')\left\{\begin{aligned}
u_{31} = {} & w + \frac{\lambda}{2(\lambda+2\mu)}\,x_3^2\,\Delta w + \mathcal{O}(\frac{h^5}{D^4}\,\varepsilon_1) \\
u_{32} = {} & x_3\left\{-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi - \frac{1}{2}\,w_{,\gamma}w_{,\gamma} + (\frac{1}{3}\,C_7x_3^2 + C_{41}h^2)w_{,\gamma\nu}w_{,\gamma\nu}\right. \\
& \left. + (C_{44}x_3^2+C_{42}h^2)(\Delta w)^2 + (C_{45}x_3^2+C_{46}h^2)w_{,\gamma}\Delta w_{,\gamma}\right\} + \mathcal{O}(\frac{h^5}{D^4}\,\bar{\varepsilon}_2)
\end{aligned}\right.$$

$$
\left\{
\begin{aligned}
e_{\alpha\beta 1} &= x_3\left\{-w_{,\alpha\beta} + \left[\frac{3\lambda+4\mu}{6(\lambda+2\mu)}\, x_3^2 - \frac{2(\lambda+\mu)}{\lambda+2\mu}\, h^2\right]\Delta w_{,\alpha\beta}\right\} + \mathcal{O}\left(\frac{h^4}{D^4}\, \varepsilon_1\right) \\
e_{\alpha\beta 2} &= \delta_{\alpha\beta}\left\{-\frac{\lambda}{3\lambda+2\mu}\, \Delta\psi + h^2\left[C_{31} w_{,\gamma\nu} w_{,\gamma\nu} + C_{32}(\Delta w)^2\right]\right\} \\
&\quad + T_{\alpha\beta} + \frac{\lambda}{2(3\lambda+2\mu)}\, \left(x_3^2 - \frac{h^2}{3}\right)\Delta\psi_{,\alpha\beta} \\
&\quad - \left[\frac{\lambda}{2(\lambda+2\mu)}\, x_3^2 + C_{23}h^2\right] w_{,\alpha\beta}\Delta w + \\
&\quad + \left(\frac{1}{2}\, x_3^2 - C_{24}h^2\right) w_{,\alpha\gamma} w_{,\gamma\beta} + \mathcal{O}\left(\frac{h^4}{D^4}\, \bar{\varepsilon}_2\right)
\end{aligned}
\right. \tag{0.16'}
$$

$$
\left\{
\begin{aligned}
e_{\alpha 3 1} &= \frac{\lambda+\mu}{\lambda+2\mu}\, (x_3^2 - h^2)\Delta w_{,\alpha} + \mathcal{O}\left(\frac{h^5}{D^5}\, \varepsilon_1\right) \\
e_{\alpha 3 2} &= x_3(x_3^2-h^2)\, \left\{\frac{C_{165}}{2\mu}\, w_{,\gamma\nu} w_{,\gamma\nu\alpha} + C_{171}\Delta w\, \Delta w_{,\alpha}\right. \\
&\quad \left. + C_{172} w_{,\alpha\gamma}\Delta w_{,\gamma}\right\} + \mathcal{O}\left(\frac{h^5}{D^5}\, \bar{\varepsilon}_2\right)
\end{aligned}
\right. \tag{0.17'}
$$

$$
\left\{
\begin{aligned}
e_{331} &= \frac{\lambda}{\lambda+2\mu}\, x_3\Delta w + \mathcal{O}\left(\frac{h^4}{D^4}\, \varepsilon_1\right) \\
e_{332} &= -\frac{\lambda}{3\lambda+2\mu}\, \Delta\psi + (C_7x_3^2 + C_{41}h^2) w_{,\gamma\nu} w_{,\gamma\nu} \\
&\quad + (C_8x_3^2 + C_{42}h^2)(\Delta w)^2 + \mathcal{O}\left(\frac{h^4}{D^4}\, \bar{\varepsilon}_2\right)
\end{aligned}
\right. \tag{0.18'}
$$

$$
(0.19')\left\{
\begin{aligned}
t_{\alpha\beta 1} &= -\frac{2\lambda\mu}{\lambda+2\mu}\, x_3\delta_{\alpha\beta}\Delta w - 2\mu x_3 w_{,\alpha\beta} \\
&\quad + \left[\frac{\mu(3\lambda+4\mu)}{3(\lambda+2\mu)}\, x_3^3 - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\, x_3 h^2\right]\Delta w_{,\alpha\beta} + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\varepsilon_1\right) \\
t_{\alpha\beta 2} &= \delta_{\alpha\beta}\left[(C_{12}x_3^2 + C_{35}h^2)w_{,\gamma\nu}w_{,\gamma\nu} + (C_{13}x_3^2 + C_{36}h^2)(\Delta w)^2\right] \\
&\quad + 2\mu T_{\alpha\beta} + \frac{\lambda\mu}{3\lambda+2\mu}\left(x_3^2 - \frac{h^2}{3}\right)\Delta\psi_{,\alpha\beta} \\
&\quad + (C_{15}x_3^2 - 2\mu C_{23}h^2)w_{,\alpha\beta}\Delta w + (C_{16}x_3^2 - 2\mu C_{24}h^2)w_{,\alpha\gamma}w_{,\gamma\beta} \\
&\quad + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\bar{\varepsilon}_2\right)
\end{aligned}
\right.
$$

$$
(0.20')\left\{
\begin{aligned}
t_{\alpha 3 1} &= \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(x_3^2 - h^2)\Delta w_{,\alpha} + \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\varepsilon_1\right) \\
t_{\alpha 3 2} &= x_3(x_3^2-h^2)\left\{C_{165}w_{,\gamma\nu}w_{,\gamma\nu\alpha} + C_{167}\Delta w\Delta w_{,\alpha} + C_{170}w_{,\alpha\gamma}\Delta w_{,\gamma}\right\} \\
&\quad + \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\bar{\varepsilon}_2\right)
\end{aligned}
\right.
$$

$$
(0.21')\left\{
\begin{aligned}
t_{331} &= \mu x_3\left(1 - \frac{x_3^2}{h^2}\right)T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{\mu h^6}{D^6}\,\varepsilon_1\right) \\
t_{332} &= (x_3^2-h^2)\Big\{\mu w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda\mu}{\lambda+2\mu}\,(\Delta w)^2 \\
&\quad + (C_{190}h^2 + C_{186}x_3^2)w_{,\gamma\nu}\Delta w_{,\gamma\nu} \\
&\quad + (x_3^2-h^2)\left(C_{188}\Delta w_{,\gamma}\Delta w_{,\gamma} - \frac{1}{4}\,C_{165}w_{,\gamma\nu\omega}w_{,\gamma\nu\omega}\right)\Big\} \\
&\quad + \mathcal{O}\left(\frac{\mu h^6}{D^6}\,\bar{\varepsilon}_2\right)
\end{aligned}
\right.
$$

$$
(0.22')\left\{\begin{aligned}
\Delta^2\psi &= \frac{3\lambda+2\mu}{4(\lambda+\mu)}\left[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2\right] \\
&\quad + h^2\left(C_{63}w_{,\gamma\nu\omega}w_{,\gamma\nu\omega} + C_{64}w_{,\gamma\nu}\Delta w_{,\gamma\nu} + C_{65}\Delta w_{,\gamma}\Delta w_{,\gamma}\right) \\
&\quad + \mathcal{O}\left(\frac{h^4}{D^6}\bar{\varepsilon}_2\right) \\
\frac{2(\lambda+\mu)}{3(\lambda+2\mu)}\,h^2\Delta^2 w &= T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{h^5}{D^6}\varepsilon_1\right) .
\end{aligned}\right.
$$

The most important of these results is perhaps (0.22'). We see that one of these equations is a von Karman-Föppl equation and the other is a von Karman-Föppl equation plus several bending terms.

A consequence of using the two strain parameters $\varepsilon_1, \varepsilon_2$ is that certain vacuities are avoided which occur if the one strain parameter $\varepsilon = \max|e_{ij}|$ is used. If the plate is slightly buckled, $\varepsilon > \varepsilon_c > 0$ where ε_c is the lowest critical value of ε, but ε_1 may be as close to zero as one pleases. Consequently, if the remainders in our results were all given with ε_1, $\bar{\varepsilon}_2$ replaced by ε, the remainder term would be the largest term in half the results when ε_1 is sufficiently small and $\varepsilon > \varepsilon_c$.

Additional work needs to be done to compare (0.22') or other simplifications of (0.22) with plate equations obtained by others. The techniques used to obtain these results are essentially the same as those used by F. John [1], and this author hereby expresses his appreciation to Professors F. John and J. Stoker for helpful discussions on the material presented in this report.

1. Splitting Functions into the Odd and Even Parts

Since separate estimates are to be found for the odd and even parts (as functions of x_3) of each function to be estimated, we are faced with the problem of finding expressions for the odd and even parts of nonlinear functions of the stresses, strains, and displacements. We first consider the strain invariants of (0.8). Letting s_{ie} be the even part and s_{io} be the odd part of s_i , we find that

$$
(1.1)\qquad
\begin{aligned}
s_{1e} &= e_{ii2} \\
s_{io} &= e_{ii1} \\
s_{2e} &= e_{ij1}e_{ij1} + e_{ij2}e_{ij2} \\
s_{2o} &= 2e_{ij1}e_{ij2} \\
s_{3e} &= 3e_{ij1}e_{jk1}e_{ki2} + e_{ij2}e_{jk2}e_{ki2} \\
s_{3o} &= e_{ij1}e_{jk1}e_{ki1} + 3e_{ij2}e_{jk1}e_{ki2}
\end{aligned}
$$

Following the above procedure, the subscript e will always denote the even part and the subscript o will always denote the odd part of a function of x_3.

Next let $\mathcal{A}_k$ $(k = 1,2,3\ldots)$ denote any analytic function of the strain invariants which is of order k in the strains e_{ij}. Then $\mathcal{A}_{ke}$ and $\mathcal{A}_{ko}$ are of course the even and odd parts of $\mathcal{A}_k$.

If $S_1, S_2, \ldots, S_n$ denote any sets of functions, then $S_1^{k_1}\ S_2^{k_2}\ \ldots\ S_n^{k_n}\ F(S_1,S_2,\ldots,S_n)$ will denote a linear combination of terms which are a product of k_1 elements of S_1 , k_2 elements of S_2 , ..., k_n elements of S_n , and a function of the elements of $S_1,S_2,\ldots S_n$. We will use notation such as $(S_1^{k_1}S_2^{k_2}+S_1^{k_3}S_2^{k_4})F(S_1,S_2)$ for a function which is a linear combination of functions of types $S_1^{k_1}S_2^{k_2}\ F(S_1,S_2)$ and $S_1^{k_3}S_2^{k_4}\ F(S_1,S_2)$. Coefficients in the various linear combinations will depend only on the elastic parameters. Notice that meaning has not been given to F itself.

Let e_1 be the set of functions e_{ij1} and e_2 be the set of functions e_{ij2}. Then $s_{io} = e_1F(e_1,e_2)$ $(i=1,2,3)$ from (1.1), and the same will be true for any analytic function of s_1,s_2,s_3. Thus $\mathcal{A}_{0e} = F(e_1,e_2)$ and $\mathcal{A}_{0o} = e_1F(e_1,e_2)$.

Since $\mathcal{A}_1 = \mathcal{A}_0s_1 + \mathcal{A}_0s_2 + \mathcal{A}_0s_3$, the above and (1.1) yields $\mathcal{A}_{1e} = (e_2+e_1^2)F(e_1,e_2)$ and $\mathcal{A}_{1o} = e_1F(e_1,e_2)$.

Since $\mathcal{A}_2 = \mathcal{A}_0s_3 + \mathcal{A}_0s_2 + \mathcal{A}_1s_1$, the above and (1.1) yields $\mathcal{A}_{2e} = (e_1^2+e_2^2)F(e_1,e_2)$ and $\mathcal{A}_{2o} = (e_1e_2+e_1^3)F(e_1,e_2)$.

Similarly $\mathcal{A}_3 = \mathcal{A}_0s_3 + \mathcal{A}_1s_2 + \mathcal{A}_2s_1$ so that $\mathcal{A}_{3e} = (e_1^2e_2 + e_2^3 + e_1^4)F(e_1,e_2)$ and $\mathcal{A}_{3o} = (e_1^3 + e_1e_2^2)F(e_1,e_2)$. Also $\mathcal{A}_4 = \mathcal{A}_1s_3 + \mathcal{A}_2s_2 + \mathcal{A}_3s_1$ so that $\mathcal{A}_{4e} = (e_1^4 + e_1^2e_2^2 + e_2^4)F(e_1,e_2)$ and $\mathcal{A}_{4o} = (e_1^3e_2 + e_1e_2^3 + e_1^5)F(e_1,e_2)$.

We list these results as (1.2).

$$
\begin{aligned}
\mathcal{A}_{0e} &= F(e_1,e_2)\\
\mathcal{A}_{0o} &= e_1F(e_1,e_2)\\
\mathcal{A}_{1e} &= (e_2+e_1^2)F(e_1,e_2)\\
\mathcal{A}_{1o} &= e_1F(e_1,e_3)\\
\mathcal{A}_{2e} &= (e_1^2 + e_2^2)F(e_1,e_2)\\
\mathcal{A}_{2o} &= (e_1e_2 + e_1^3)F(e_1,e_2)\\
\mathcal{A}_{3e} &= (e_1^2e_2 + e_2^3 + e_1^4)F(e_1,e_2)\\
\mathcal{A}_{3o} &= (e_1^3 + e_1e_2^2)F(e_1,e_2)\\
\mathcal{A}_{4e} &= (e_1^4 + e_1^2e_2^2 + e_2^4)F(e_1,e_2)\\
\mathcal{A}_{4o} &= (e_1^3e_2 + e_1e_2^3 + e_1^5)F(e_1,e_2)
\end{aligned}
\tag{1.2}
$$

From (0.9), (0.2), (0.8) and the fact that W is second order in the strains,

$$(1.3)\begin{cases} W = \frac{\lambda}{2} s_1^2 + \mu s_2 + \frac{A}{3} s_1^3 + B s_1 s_2 + \frac{C}{3} s_3 + \mathcal{A}_4 \\ t_{ij} = (\lambda s_1 + A s_1^2 + B s_2 + \mathcal{A}_3)\delta_{ij} + 2(\mu + B s_1 + \mathcal{A}_2) e_{ij} + (C + \mathcal{A}_1) e_{ik} e_{kj}. \end{cases}$$

From (1.3), (1.2), and (1.1)

$$(1.4)\begin{cases} W_e = \frac{\lambda}{2} (s_{1e}^2 + s_{1o}^2) + \mu s_{2e} + \frac{A}{3} (s_{1e}^3 + 3 s_{1e} s_{1o}^2) + B(s_{1e} s_{2e} + s_{1o} s_{2o}) \\ \qquad + \frac{C}{3} s_{3e} + (e_1^4 + e_1^2 e_2^2 + e_2^4) F(e_1, e_2) \\ \quad = \frac{\lambda}{2} (s_{1e}^2 + s_{1o}^2) + \mu s_{2e} + (e_1^2 e_2 + e_2^3 + e_1^4) F(e_1, e_2) = (e_1^2 + e_2^2) F(e_1, e_2) \\ W_o = \lambda s_{1e} s_{1o} + \mu s_{2o} + \frac{A}{3} (s_{1o}^3 + 3 s_{1o} s_{1e}^2) + B(s_{1e} s_{2o} + s_{1o} s_{2e}) \\ \qquad + \frac{C}{3} s_{3o} + (e_1^3 e_2 + e_1 e_2^3 + e_1^5) F(e_1, e_2) \\ \quad = \lambda s_{1e} s_{1o} + \mu s_{2o} + (e_1^3 + e_1 e_2^2) F(e_1, e_2) = (e_1 e_2 + e_1^3) F(e_1, e_2) \\ t_{ij1} = (\lambda s_{1o} + 2A s_{1e} s_{1o} + B s_{2o}) \delta_{ij} + 2(\mu + B s_{1e}) e_{ij1} + 2B s_{1o} e_{ij2} \\ \qquad + C(e_{ik1} e_{kj2} + e_{ik2} e_{kj1}) + (e_1^3 + e_1 e_2^2) F(e_1, e_2) \\ \quad = \lambda e_{kk1} \delta_{ij} + 2\mu e_{ij1} + (e_1 e_2 + e_1^3) F(e_1, e_2) \\ t_{ij2} = [\lambda s_{1e} + A(s_{1e}^2 + s_{1o}^2) + B s_{2e}] \delta_{ij} + 2(\mu + B s_{1e}) e_{ij2} + 2B s_{1o} e_{ij1} \\ \qquad + C(e_{ik1} e_{kj1} + e_{ik2} e_{kj2}) + (e_1^2 e_2 + e_2^3 + e_1^4) F(e_1, e_2) \\ \quad = \lambda e_{kk2} \delta_{ij} + 2\mu e_{ij2} + (e_1^2 + e_2^2) F(e_1, e_2) \end{cases}$$

The equations for $t_{ij\omega}$ in (1.4) can be inverted when the strains are sufficiently near zero (i.e. when θ is sufficiently small) to obtain

$$\begin{aligned} e_{ij1} &= \frac{1}{2\mu}\left(t_{ij1} - \frac{\lambda}{3\lambda+2\mu}\, t_{kk1}\delta_{ij}\right) + (t_1t_2+t_1^3)F(t_1,t_2) = t_1F(t_1,t_2) \\ e_{ij2} &= \frac{1}{2\mu}\left(t_{ij2} - \frac{\lambda}{3\lambda+2\mu}\, t_{kk2}\delta_{ij}\right) + (t_1^2+t_2^2)F(t_1,t_2) \\ &= (t_2 + t_1^2)F(t_1,t_2) \end{aligned} \tag{1.5}$$

where t_ω is the set of functions $t_{ij\omega}$ $(\omega=1,2)$.

We introduce p_{ij} in (1.6) and define $p_{ij\omega}$ $(\omega=1,2)$ in the usual way. Then the rest of (1.6) follows from (1.4), (1.5), (1,3), (0.3) and (0.4).

$$\left\{\begin{aligned} p_{ij} &= 2e_{ik}t_{kj} - W\delta_{ij} \\ p_{ij1} &= 2e_{ik1}t_{kj2} + 2e_{ik2}t_{kj1} - W_o\delta_{ij} = (t_1t_2+t_1^3)F(t_1,t_2) \\ p_{ij2} &= 2e_{ik1}t_{kj1} + 2e_{ik2}t_{kj2} - W_e\delta_{ij} = (t_1^2+t_2^2)F(t_1,t_2) \\ p_{ij\omega} &= p_{ji\omega} \\ p_{\alpha 3\omega} &= 0 \quad \text{for} \quad x_3 = \pm h \\ t_{ij\omega,j} &+ p_{ij\omega,j} = 0 \end{aligned}\right. \tag{1.6}$$

Lines 1, 2, and 4 of (1.7) are definitions, and the fifth line contains the compatibility conditions for the strains. (g^{ij}) denotes a matrix and $(\delta_{ij}+ 2e_{ij})^{-1}$ denotes the inverse of the matrix $(\delta_{ij}+2e_{ij})$. The third line follows from the second line when the strains are sufficiently near zero.

$$(1.7)\quad\left\{\begin{aligned}
\Gamma_{ijk} &= e_{jk,i} + e_{ik,j} - e_{ij,k} \\
(g^{ij}) &= (\delta_{ij} + 2e_{ij})^{-1} \\
g^{ij} &= \delta_{ij} - 2e_{ij} + 4e_{ik}e_{kj} - 8e_{ik}e_{k\ell}e_{\ell j} + \cdots \\
c_{ijk\ell} &= \Gamma_{ijn}\, g^{nm}\, \Gamma_{k\ell m} - \Gamma_{ikn}\, g^{nm}\, \Gamma_{j\ell m} \\
e_{j\ell,ik} &+ e_{ik,j\ell} - e_{ij,k\ell} - e_{k\ell,ij} = c_{ijk\ell}
\end{aligned}\right.$$

We introduce $\Gamma_{ijk\omega}$, $g^{ij}{}_{\omega}$, and $c_{ijk\ell\omega}$ $(\omega=1,2)$ in the usual way. For example, $c_{ijk\ell} = c_{ijk\ell 1} + c_{ijk\ell 2}$ where $c_{ijk\ell\omega}$ is an even function of x_3 if ω plus the number of threes among i,j,k,ℓ is even, and otherwise $c_{ijk\ell\omega}$ is an odd function of x_3. Then we obtain (1.8) from (1.7) and (1.5).

$$(1.8)\quad\left\{\begin{aligned}
\Gamma_{ijk\omega} &= e_{jk\omega,i} + e_{ik\omega,j} - e_{ij\omega,k} \\
g^{ij}{}_{1} &= e_1 F(e_1,e_2) = t_1 F(t_1,t_2) \\
g^{ij}{}_{2} &= \delta_{ij} + F(e_1,e_2) = \delta_{ij} + F(t_1,t_2) \\
c_{ijk\ell 1} &= \Gamma_{ijn1}(g^{nm}{}_{2}\, \Gamma_{k\ell m2} + g^{nm}{}_{1}\, \Gamma_{k\ell m1}) \\
&\quad + \Gamma_{ijn2}(g^{nm}{}_{1}\, \Gamma_{k\ell m2} + g^{nm}{}_{2}\, \Gamma_{k\ell m1}) \\
&\quad - \Gamma_{ikn1}(g^{nm}{}_{2}\, \Gamma_{j\ell m2} + g^{nm}{}_{1}\, \Gamma_{j\ell m1}) \\
&\quad - \Gamma_{ikn2}(g^{nm}{}_{1}\, \Gamma_{j\ell m2} + g^{nm}{}_{2}\, \Gamma_{j\ell m1}) \\
c_{ijk\ell 2} &= \Gamma_{ijn1}(g^{nm}{}_{1}\, \Gamma_{k\ell m2} + g^{nm}{}_{2}\, \Gamma_{k\ell m1}) \\
&\quad + \Gamma_{ijn2}(g^{nm}{}_{1}\, \Gamma_{k\ell m1} + g^{nm}{}_{2}\, \Gamma_{k\ell m2}) \\
&\quad - \Gamma_{ikn1}(g^{nm}{}_{1}\, \Gamma_{j\ell m2} + g^{nm}{}_{2}\, \Gamma_{j\ell m1}) \\
&\quad - \Gamma_{ikn2}(g^{nm}{}_{1}\, \Gamma_{j\ell m1} + g^{nm}{}_{2}\, \Gamma_{j\ell m2}) \\
e_{j\ell\omega,ik} &+ e_{ik\omega,j\ell} - e_{ij\omega,k\ell} - e_{k\ell\omega,ij} = c_{ijk\ell\omega}
\end{aligned}\right.$$

If A is a set of functions of x_1, x_2, x_3, we let A' denote the set of all first derivatives of elements of A. Since A' is also a set of functions, this also defines A", A''', etc.

From (1.5) and (1.8) we obtain

$$(1.9)\quad \begin{cases} e_{ij1,k} = (t_1' + t_1 t_2')F(t_1,t_2) \\ e_{ij2,k} = (t_2' + t_1 t_1' + t_2 t_1)F(t_1,t_2) \\ c_{ijk\ell 1} = (t_1' t_2' + t_1 t_2'^2 + t_1 t_1'^2 + t_2 t_1'^2)F(t_1,t_2) \\ c_{ijk\ell 2} = (t_1 t_1' t_2' + t_1'^2 + t_2 t_1' t_2' + t_2'^2)F(t_1,t_2) \ . \end{cases}$$

We saw above that a derivative of a function of type $t_1F(t_1,t_2)$ is a function of type $(t_1'+t_1t_2')F(t_1,t_2)$. We therefore let $[t_1F(t_1,t_2)]'$ denote a function of type $(t_1' + t_1t_2')F(t_1,t_2)$. We do this even if the function of type $(t_1' + t_1t_2')F(t_1,t_2)$ is not the derivative of a function of type $t_1F(t_1,t_2)$. Similarly $[t_1F(t_1,t_2)]''$ denotes a function of type $(t_1'' + t_1t_2'' + t_1't_2' + t_1'^2 + t_1t_2'^2)F(t_1,t_2)$, etc. Using this convention, $[(t_1t_2+t_1^3)F(t_1,t_2)]'$ denotes a function of type $(t_1't_2+t_1t_2'+t_1^2t_1')F(t_1,t_2)$ and $[(t_1t_2+t_1^3)F(t_1,t_2)]''$ denotes a function of type $(t_1''t_2+t_1t_2'' +t_1^2t_1''+t_1't_2'+t_1t_1'^2+t_2t_1'^2+t_1t_2'^2)F(t_1,t_2)$. Comparing this with (1.9), we get the first line of (1.10). Similarly, we get the second line of (1.10). This makes it possible for us to combine $c_{ijk\ell\omega}$ with other functions which occur later in (1.11).

$$(1.10)\quad \begin{aligned} c_{ijk\ell 1} &= [(t_1t_2+t_1^3)F(t_1,t_2)]'' \\ c_{ijk\ell 2} &= [(t_1^2+t_2^2)\ F(t_1,t_2)]'' \end{aligned}$$

Using (1.5), (1.10), and the compatibility conditions from (1.8), we obtain

$$
(1.11)\left\{\begin{aligned}
& t_{j\ell 1,ik} + t_{ik1,j\ell} - t_{ij1,k\ell} - t_{k\ell 1,ij} \\
& = \frac{\lambda}{3\lambda+2\mu}\,(t_{mm1,ik}\delta_{j\ell} + t_{mm1,j\ell}\delta_{ik} - t_{mm1,k\ell}\delta_{ij} - t_{mm1,ij}\delta_{k\ell}) \\
& \qquad + [(t_1 t_2 + t_1^3)F(t_1,t_2)]'' \\
& t_{j\ell 2,ik} + t_{ik2,j\ell} - t_{ij2,k\ell} - t_{k\ell 2,ij} \\
& = \frac{\lambda}{3\lambda+2\mu}\,(t_{mm2,ik}\delta_{j\ell} + t_{mm2,j\ell}\delta_{ik} - t_{mm2,k\ell}\delta_{ij} - t_{mm2,ij}\delta_{k\ell}) \\
& \qquad + [(t_1^2 + t_2^2)F(t_1,t_2)]''
\end{aligned}\right.
$$

2. Relations Among the Derivatives of the Stresses.

If S is a set of functions of x_1, x_2, x_3, we let $\dot{S}$ be the set of first derivatives with respect to x_1, x_2 of functions in S. Since $\dot{S}$ is also a set of functions, we have defined $\ddot{S}$, $\dot{S}'$, etc. We let ${}^{\cdot(k)}$ denote k super-dots and ${}^{\prime(k)}$ denote k super primes. If S, S_1, S_2 denote sets of functions, notation such as

$$S = S_1 + S_2 + (t_1 t_2 + t_1^3) F(t_1, t_2)$$

means each function in S is a linear combination of functions in S_1 and S_2 and a function of type $(t_1 t_2 + t_1^3) F(t_1, t_2)$. The coefficients in these linear combinations will depend only on the elastic constants.

Let $a_{k\omega}$ $(k=1,2,\ldots;\ \omega=1,2)$ be the set of functions of the type

$$\frac{\partial^k t_{ij\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

such that $k_3 = 0$; or $k_3 = 1$ and $(i,j) = (\alpha,3)$ or $(3,3)$; or k_3 is even and $(i,j) = (\alpha,\beta)$ or $(3,3)$; or k_3 is odd and $(i,j) = (\alpha, 3)$. The various cases are not mutually exclusive.

Let $b_{k\omega}$ $(k=1,2,\ldots;\ \omega=1,2)$ be the set of all functions

$$\frac{\partial^k t_{ij\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

with no restrictions.

We next show that

$$(2.1)\quad \begin{cases} a_{k1} = \dot{t}_1^{(k)} + [(t_1t_2+t_1^3)F(t_1,t_2)]'^{(k)} \\ a_{k2} = \dot{t}_2^{(k)} + [(t_1^2+t_2^2)F(t_1,t_2)]'^{(k)} \\ b_{k1} = \{\dot{t}^{(k-1)}_{\alpha\beta 1,3}\} + \dot{t}_1^{(k)} + [(t_1t_2+t_1^3)F(t_1,t_2)]'^{(k)} \\ b_{k2} = \{\dot{t}^{(k-1)}_{\alpha\beta 2,3}\} + \dot{t}_2^{(k)} + [(t_1^2+t_2^2)F(t_1,t_2)]'^{(k)} \end{cases}$$

for $k = 1,2,3,\ldots$.

Here $\{\dot{t}^{(k-1)}_{\alpha\beta\omega,3}\}$ denotes the set of all functions of the type

$$\frac{\partial^k t_{\alpha\beta\omega}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3} \quad .$$

The functions in $a_{1\omega}$ are those of the types $t_{ij\omega,\alpha}$ and $t_{i3\omega,3}$. The conclusion for functions in $a_{1\omega}$ follows immediately from the following results listed in (1.6).

$$(2.2)\quad \begin{cases} t_{i3\omega,3} = -t_{i\alpha\omega,\alpha} - p_{ij\omega,j} \\ p_{ij1} = (t_1t_2+t_1^3)F(t_1,t_2) \\ p_{ij2} = (t_1^2+t_2^2)F(t_1,t_2) \quad . \end{cases}$$

The only functions in $b_{1\omega}$ which are not in $a_{1\omega}$ are of the type $t_{\alpha\beta\omega,3}$. Thus the conclusion is trivial for $b_{1\omega}$.

Each function in $a_{2\omega}$ $(b_{2\omega})$ is the derivative with respect to x_1 or x_2 of a function in $a_{1\omega}$ $(b_{1\omega})$ unless it is of the form $t_{ij\omega,33}$. Thus (2.1) is valid for those functions of $a_{2\omega}$ and $b_{2\omega}$ for which not all differentiations are with respect to x_3.

From (2.2) we have

$$t_{\beta 31,33} = \{\dot{t}_{\alpha\gamma 1,3}\} + [(t_1t_2+t_1^3)F(t_1,t_2)]''$$

$$t_{\beta 32,33} = \{\dot{t}_{\alpha\gamma 2,3}\} + [(t_1^2+t_2^2)F(t_1,t_2)]''$$

$$t_{331,33} = -t_{\alpha 31,\alpha 3} + [(t_1t_2+t_1^3)F(t_1,t_2)]''$$

$$t_{332,33} = -t_{\alpha 32,\alpha 3} + [(t_1^2+t_2^2)F(t_1,t_2)]'' \ .$$

Since we already know (2.1) is valid for the functions $t_{\alpha 3\omega,\beta 3}$, the above shows (2.1) is valid for the functions $t_{i3\omega,33}$. This takes care of all functions in $a_{2\omega}$ and $b_{2\omega}$ with the exception of the functions $t_{\alpha\beta\omega,33}$ which are in both $a_{2\omega}$ and $b_{2\omega}$.

Setting $k = \ell = 3$ and $j = i$ and summing over i in (1.11), we obtain

$$t_{mm1,33} = \frac{\lambda}{\lambda+2\mu}\, t_{mm1,\gamma\gamma} + \frac{3\lambda+2\mu}{\lambda+2\mu}\,(2t_{i31,i3} - t_{331,ii})$$
$$+ [(t_1t_2+t_1^3)F(t_1,t_2)]'' \ .$$

But $t_{i31,i3} = -\,p_{i31,i3} = [(t_1t_2+t_1^3)F(t_1,t_2)]''$ from (1.6). Thus we have the first line of (2.3). Similarly, the second line is obtained.

$$(2.3)\begin{cases} t_{mm1,33} = \dfrac{\lambda}{\lambda+2\mu}\, t_{mm1,\gamma\gamma} - \dfrac{3\lambda+2\mu}{\lambda+2\mu}\, t_{331,ii} + [(t_1t_2+t_1^3)F(t_1,t_2)]'' \\[2ex] t_{mm2,33} = \dfrac{\lambda}{\lambda+2\mu}\, t_{mm2,\gamma\gamma} - \dfrac{3\lambda+2\mu}{\lambda+2\mu}\, t_{331,ii} + [(t_1^2+t_2^2)F(t_1,t_2)]''. \end{cases}$$

Setting $i = \alpha$, $j = \beta$, $k = \ell = 3$ in (1.11), we obtain

$$t_{\alpha\beta 1,33} = t_{\alpha 31,\beta 3} + t_{\beta 31,\alpha 3} - t_{331,\alpha\beta}$$
$$+ \frac{\lambda}{3\lambda+2\mu}\,(t_{mm1,\alpha\beta} + t_{mm1,33}\delta_{\alpha\beta}) + [(t_1t_2+t_1^3)F(t_1,t_2)]''.$$

Using the first line of (2.3), this becomes the first line of (2.4). Similarly, we obtain the second line of (2.4).

$$(2.4)\begin{cases} t_{\alpha\beta 1,33} = t_{\alpha 31,\beta 3} + t_{\beta 31,\alpha 3} - t_{331,\alpha\beta} + \frac{\lambda}{3\lambda+2\mu} t_{mm1,\alpha\beta} \\ \qquad + \frac{\lambda^2}{(\lambda+2\mu)(3\lambda+2\mu)} \delta_{\alpha\beta} t_{mm1,\gamma\gamma} - \frac{\lambda}{\lambda+2\mu} \delta_{\alpha\beta} t_{331,ii} \\ \qquad + [(t_1 t_2 + t_1^3) F(t_1,t_2)]'' \\ t_{\alpha\beta 2,33} = t_{\alpha 32,\beta 3} + t_{\beta 32,\alpha 3} - t_{332,\alpha\beta} + \frac{\lambda}{3\lambda+2\mu} t_{mm2,\alpha\beta} \\ \qquad + \frac{\lambda^2}{(\lambda+2\mu)(3\lambda+2\mu)} \delta_{\alpha\beta} t_{mm2,\gamma\gamma} - \frac{\lambda}{\lambda+2\mu} \delta_{\alpha\beta} t_{332,ii} \\ \qquad + [(t_1^2 + t_2^2) F(t_1,t_2)]'' \end{cases}$$

(2.4) tells us that $t_{\alpha\beta\omega,33}$ is a linear combination of functions of $a_{2\omega}$ and a remainder of the appropriate type. The functions $a_{2\omega}$ involved are those for which we already know (2.1) holds. This completes the proof of (2.1) for $k = 2$.

For $k \geq 3$, (2.1) follows by induction. If it is true for some k, each function in $a_{(k+1)\omega}$ $(b_{(k+1)\omega})$ is the derivative with respect to x_1 or x_2 of a function in $a_{k\omega}$ $(b_{k\omega})$ unless it is of the form $\partial^{k+1} t_{ij\omega}/\partial x_3^{k+1}$. Thus (2.1) is valid for those functions of $a_{(k+1)\omega}$ and $b_{(k+1)\omega}$ for which not all differentations are with respect to x_3. We then obtain (2.1) for functions of the type $\partial^{k+1} t_{ij\omega}/\partial x_3^{k+1}$ by differentiating (2.2) and (2.4) with respect to x_3 the appropriate number of times.

3. Proof of (0.5), (0.6), and (0.7).

Let

$$\tag{3.1} \begin{cases} r = \sqrt{(x_1-\hat{x}_1)^2+(x_2-\hat{x}_2)^2} \\[2ex] \zeta_k(r) = \left\{\begin{array}{lll} \left(1-\dfrac{r^2}{\theta^2D^2}\right)^2 & \text{for} & 0 \le r \le \theta D \\ 0 & \text{for} & r \ge \theta D \end{array}\right\} \quad \text{for } k = 0,1,2 \\[4ex] \zeta_k(r) = \left\{\begin{array}{lll} \left(1-\dfrac{2^{2k-4}\,r^2}{\theta^2D^2}\right)^2 & \text{for} & 0 \le r \le \dfrac{\theta D}{2^{k-2}} \\ 0 & \text{for} & r \ge \theta D/2^{k-2} \end{array}\right\} \quad \text{for } k = 3,4,\ldots \end{cases}$$

where $(\hat{x}_1,\hat{x}_2,0)$ is any interior point such that the unstrained distance to the edge of the plate is at least θD. Then ζ_k is continuous and has continuous first derivatives everywhere.

For a finite set A of functions of x_1,x_2,x_3, let $|A|^2$ be the sum of the squares of the functions in A and let $|A| \ge 0$. If we let ζ_k also denote the set whose one function is the one defined in (3.1), then $\dot{\zeta}_k$, $\ddot{\zeta}_k$, $|\dot{\zeta}_k|$, and $|\ddot{\zeta}_k|$ are all defined, and it follows easily that

$$\tag{3.2} \begin{cases} |\dot{\zeta}_k| = \mathcal{O}\left(\dfrac{1}{\theta D}\sqrt{\zeta_k}\right) = \mathcal{O}\left(\dfrac{1}{\theta D}\right) \\[3ex] |\ddot{\zeta}_k| = \mathcal{O}\left(\dfrac{1}{\theta^2D^2}\right) . \end{cases}$$

In the introduction we said that the constant, whose existence is claimed in an order relation, may depend on the material and on θ. However, *in this section it will be possible to choose all such constants to be independent of* θ. To accomplish this we must show necessary dependence on θ explicitly.

Let $\|A\|_k^2 = \iiint |A|^2\, dx_1\, dx_2\, dx_3$ and $\|\zeta_k A\|_k^2 = \iiint \zeta_k^2 |A|^2 dx_1 dx_2 dx_3$ where A is a finite set of functions and the integration is over the region where $|x_3| \leq h$ and $\zeta_k \neq 0$. Let $B = \mathcal{O}(C) + o(D)$ mean that for each constant $k > 0$ there is a $K > 0$ depending only on the material and independent of θ such that $|B| \leq K|C| + k|D|$. Then from the Appendix of the first paper of [1], we obtain

$$
(3.3)\begin{cases}
\|\zeta_k \dot{A}\|_k = \mathcal{O}\left(\frac{1}{\theta D} \|A\|_k\right) + o\left(\theta D \|\zeta_k \ddot{A}\|_k\right) \\
\|\zeta_k A_{,3}\|_k = \mathcal{O}\left(\frac{1}{h} \|A\|_k\right) + o\left(h \|\zeta_k A_{,33}\|_k\right) \\
\|\zeta_k A'\|_k = \mathcal{O}\left(\frac{1}{h} \|A\|_k\right) + o\left(\theta D \|\zeta_k \ddot{A}\|_k + h \|\zeta_k A_{,33}\|_k\right) \\
\|\dot{\zeta}_k A_{,3}\|_k = \mathcal{O}\left(\frac{1}{\theta h D} \|A\|_k\right) + o\left(\frac{h}{\theta D} \|\zeta_k A_{,33}\|_k\right) \\
\|\dot{\zeta}_k \dot{A}\|_k = \mathcal{O}\left(\frac{1}{\theta^2 D^2} \|A\|_k\right) + o\left(\|\zeta_k \ddot{A}\|_k\right) \\
\|\dot{\zeta}_k A'\|_k = \mathcal{O}\left(\frac{1}{\theta h D} \|A\|_k\right) + o\left(\|\zeta_k A''\|_k\right) \\
\|\zeta_k \dot{A}^2\|_k = \mathcal{O}[\bar{A}(\|\zeta_k \ddot{A}\|_k + \frac{1}{\theta^2 D^2} \|A\|_k)] \\
\|\zeta_k A_{,3}^2\|_k = \mathcal{O}[\bar{A}(\|\zeta_k A_{,33}\|_k + \frac{1}{h^2} \|A\|_k] \\
\|\zeta_k A'^2\|_k = \mathcal{O}[\bar{A}(\|\zeta_k A''\|_k + \frac{1}{h^2} \|A\|_k)] \\
\|\zeta_k \dot{A}_{,3}\|_k = \mathcal{O}\left(\frac{1}{\theta h D} \|A\|_k + \frac{\theta D}{h} \|\zeta_k \ddot{A}\|_k + \frac{h}{\theta D} \|\zeta_k A_{,33}\|_k\right) \\
\frac{\sqrt{h}}{\theta D} \zeta_k |A| = \mathcal{O}[\|\zeta_k \ddot{A}\|_k + \frac{1}{\theta^2 D^2} \|A\|_k + \frac{h^2}{\theta^2 D^2} \|\zeta_k A''\|_k]
\end{cases}
$$

In (3.3), $\bar{A}$ is an upper bound for the absolute values of all the functions in A, $A_{,3}$ is the set of first derivatives with respect to x_3 of functions in A, $\dot{A}_{,3}$ is the set of second derivatives of functions in A with one differentiation with respect to x_1 and x_2 and the other with respect to x_3 , $A_{,33}$ is the set of second derivatives of functions in A with respect to x_3 , and notation such as $\dot{A}^2$ denotes the set of all functions which are the square of a function from $\dot{A}$.

Next we claim that

$$(3.4)\left\{\begin{aligned}
\|\zeta_k \dot{t}_1^{(k)}\|_k &= \mathcal{O}\left\{\frac{1}{\theta^2 D^2}\|\dot{t}_1^{(k-2)}\|_k + \frac{1}{\theta^2 D^2}\|[(t_1t_2+t_1^3)F(t_1,t_2)]'^{(k-2)}\|_k\right.\\
&\quad + \frac{\theta D}{h}\|\zeta_k[(t_1t_2+t_1^3)F(t_1,t_2)]'^{\cdot(k-1)}\|_k\\
&\quad \left. + \|\zeta_k[(t_1t_2+t_1^3)F(t_1,t_2)]'^{(k)}\|_k\right\} + \mathcal{O}\left[\frac{h}{\theta D}\|\zeta_k t_1'^{(k)}\|_k\right]\\
\|\zeta_k \dot{t}_2^{(k)}\|_k &= \mathcal{O}\left\{\frac{1}{\theta^2 D^2}\|\dot{t}_2^{(k-2)}\|_k + \frac{1}{\theta^2 D^2}\|[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k-2)}\|_k\right.\\
&\quad + \frac{\theta D}{h}\|\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]'^{\cdot(k-1)}\|_k\\
&\quad \left. + \|\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]'^{(k)}\|_k\right\} + \mathcal{O}\left[\frac{h}{\theta D}\|\zeta_k t_2'^{(k)}\|_k\right]
\end{aligned}\right.$$

where $'^{\cdot(k-1)}$ denotes one prime and k-1 dots. Notation such as $\|\zeta_k[(t_1t_2+t_1^3)F(t_1,t_2)]'^{(k)}\|_k^2$ denotes the integral of the sum of the squares of a specific finite set of functions each of which is ζ_k times a function of the type $[(t_1t_2+t_1^3)F(t_1,t_2)]'^{(k)}$. As usual the domain of integration is the set where $|x_3| \leq h$ and $\zeta_k \neq 0$.

We will prove (3.4) for k = 2. The same method is valid for all k. From (1.8)

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta}(2e_{i\alpha\omega,\beta j} - e_{\alpha\beta\omega,ij} - c_{ij\alpha\beta\omega}) \, dx_1 \, dx_2 \, dx_3$$

where the region of integration is such that $|x_3| \leq h$ and $\zeta_2 \neq 0$.

Using the divergence theorem, $t_{ij\omega,j} = -p_{ij\omega,j}$ from (1.6), and $t_{i3\omega} = 0$ for $x_3 = \pm h$, we have

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \iiint \left\{ \left[\zeta_2^2 \, p_{ij\omega,j\alpha\beta} - (\zeta_2^2)_{,\gamma} \, t_{i\gamma\omega,\alpha\beta} \right] \left[2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i} \right] - \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, c_{ij\alpha\beta\omega} \right\} dx_1 \, dx_2 \, dx_3 \, .$$

Similarly

$$\iiint (\zeta_2^2)_{,\gamma} \, t_{i\gamma\omega,\alpha\beta} \, e_{\alpha\beta\omega,i} \, dx_1 \, dx_2 \, dx_3$$

$$= \iiint \left[(\zeta_2^2)_{,\gamma} \, p_{i\gamma\omega,i\alpha\beta} \, e_{\alpha\beta\omega} - (\zeta_2^2)_{,\gamma\nu} \, t_{\gamma\nu\omega,\alpha\beta} e_{\alpha\beta\omega} \right] dx_1 \, dx_2 \, dx_3 \, .$$

Thus

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \iiint \Big\{ \zeta_2^2 \, p_{ij\omega,j\alpha\beta}(2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i}) - 2(\zeta_2^2)_{,\gamma} \, t_{i\gamma\omega,\alpha\beta} \, e_{i\alpha\omega,\beta} + \left[(\zeta_2^2)_{,\gamma} \, p_{i\gamma\omega,i\alpha\beta} - (\zeta_2^2)_{,\gamma\nu} \, t_{\gamma\nu\omega,\alpha\beta} \right] e_{\alpha\beta\omega} - \zeta_2^2 t_{ij\omega,\alpha\beta} c_{ij\alpha\beta\omega} \Big\} dx_1 dx_2 dx_3$$

$$= \iiint \Big\{ -\zeta_2^2 \, p_{ij\omega,j\alpha}(2e_{i\alpha\omega,\beta\beta} - e_{\alpha\beta\omega,i\beta})$$

$$- (\zeta_2^2)_{,\beta} \, p_{ij\omega,j\alpha}(2e_{i\alpha\omega,\beta} - e_{\alpha\beta\omega,i})$$

$$- 2(\zeta_2^2)_{,\gamma} \, t_{i\gamma\omega,\alpha\beta} \, e_{i\alpha\omega,\beta} - (\zeta_2^2)_{,\gamma\nu} \, t_{\gamma\nu\omega,\alpha\beta} \, e_{\alpha\beta\omega}$$

$$- (\zeta_2^2)_{,\gamma} \, p_{i\gamma\omega,i\alpha} \, e_{\alpha\beta\omega,\beta} - (\zeta_2^2)_{,\gamma\beta} \, p_{i\gamma\omega,i\alpha} \, e_{\alpha\beta\omega}$$

$$- \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, c_{ij\alpha\beta\omega} \Big\} \, dx_1 \, dx_2 \, dx_3$$

using the divergence theorem again. This yields

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \mathcal{O}\left[\frac{\theta^2 D^2}{\mu \, h^2} \|\zeta_2 \dot{p}'_\omega\|_2^2 + \mu\|\dot{\zeta}_2 \dot{e}_\omega\|_2^2 + \frac{\mu}{\theta^4 D^4} \|e_\omega\|_2^2 + \mu\|\zeta_2 c_\omega\|_2^2\right]$$

$$+ \mathcal{O}\left[\frac{\mu \, h^2}{\theta^2 D^2} (\|\zeta_2 e''_\omega\|_2^2 + \|\dot{\zeta}_2 e'_\omega\|_2^2) + \frac{1}{\mu} \|\zeta_2 \ddot{t}_\omega\|_2^2\right]$$

where we have used (3.2), $h/D \leq \theta$ and the inequality $AB \leq \frac{1}{\alpha} A^2 + \alpha B^2$ or $AB = \mathcal{O}(A^2) + \mathcal{O}(B^2)$ for all A, B, α. Also we have let c_ω denote the set of functions $c_{ijk\ell\omega}$, and p_ω the set of functions $p_{ij\omega}$.

Using lines 5 and 6 of (3.3), the above becomes

$$\iiint \zeta_2^2 \, t_{ij\omega,\alpha\beta} \, e_{ij\omega,\alpha\beta} \, dx_1 \, dx_2 \, dx_3$$

$$= \mathcal{O}\left(\frac{\theta^2 D^2}{\mu h^2} \|\zeta_2 \dot{p}'_\omega\|_2^2 + \frac{\mu}{\theta^4 D^4} \|e_\omega\|_2^2 + \mu\|\zeta_2 c_\omega\|_2^2\right)$$

$$+ \mathcal{O}\left(\frac{1}{\mu} \|\zeta_2 \ddot{t}_\omega\|_2^2 + \mu\|\zeta_2 \ddot{e}_\omega\|_2^2 + \frac{\mu h^2}{\theta^2 D^2} \|\zeta_2 e''_\omega\|_2^2\right) .$$

Next let $z_{ij\omega} = e_{ij\omega} - \frac{1}{2\mu}\left(t_{ij\omega} - \frac{\lambda}{3\lambda+2\mu} t_{kk\omega}\delta_{ij}\right)$.
Then from (1.5)

$$z_{ij1} = (t_1 t_2 + t_1^3)F(t_1,t_2)$$
$$z_{ij2} = (t_1^2 + t_2^2)F(t_1,t_2) \ .$$

Also

$$t_{ij\omega,\alpha\beta}e_{ij\omega,\alpha\beta} = \frac{1}{2\mu}\left(|\ddot{t}_\omega|^2 - \frac{\lambda}{3\lambda+2\mu} t_{kk\omega,\alpha\beta}t_{\ell\ell\omega,\alpha\beta}\right) + t_{ij\omega,\alpha\beta}z_{ij\omega,\alpha\beta}$$

$$\geq \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} |\ddot{t}_\omega|^2 + t_{ij\omega,\alpha\beta}z_{ij\omega,\alpha\beta}$$

so that

$$|\ddot{t}_\omega|^2 = \mathcal{O}\left(\mu\ t_{ij\omega,\alpha\beta}e_{ij\omega,\alpha\beta} + \mu t_{ij\omega,\alpha\beta}z_{ij\omega,\alpha\beta}\right)$$

$$= \mathcal{O}\left(\mu\ t_{ij\omega,\alpha\beta}e_{ij\omega,\alpha\beta} + \mu^2|\ddot{z}_\omega|^2\right) + \quad \left(|\ddot{t}_\omega|^2\right)$$

where z_ω denotes the set of functions $z_{ij\omega}$.

Thus

$$|\ddot{t}_\omega|^2 = \mathcal{O}\left(\mu\ t_{ij\omega,\alpha\beta}e_{ij\omega,\alpha\beta} + \mu^2|\ddot{z}_\omega|^2\right) \ , \quad \text{and}$$

$$\|\zeta_2\ddot{t}_\omega\|_2^2 = \mathcal{O}\left[\frac{\theta^2 D^2}{h^2}\|\zeta_2\dot{p}'_\omega\|_2^2 + \frac{\mu^2}{\theta^4 D^4}\|e_\omega\|_2^2\right.$$
$$\left. + \mu^2\|\zeta_2 c_\omega\|_2^2 + \mu^2\|\zeta_2\ddot{z}_\omega\|_2^2\right]$$
$$+ o\left(\|\zeta_2 t_\omega\|_2^2 + \mu^2\|\zeta_2\ddot{e}_\omega\|_2^2 + \frac{\mu^2 h^2}{\theta^2 D^2}\|\zeta_2 e''_\omega\|^2\right) \ .$$

From this, (1.5), (1.6), and (1.10), we obtain (3.4) for k = 2.

We define

(3.5) $A_{k\omega} = \|\zeta_k a_{k\omega}\|_k \ , \quad B_{k\omega} = \|\zeta_k\ b_{k\omega}\|_k \qquad (k=1,2,\ldots;\ \omega=1,2);$

where $\zeta_k a_{k\omega}$ denotes the set of functions obtained by multiplying each function of $a_{k\omega}$ by ζ_k , etc.

From (3.4) and (2.1) we obtain

$$(3.6)\left\{\begin{aligned}
A_{k1} = \mathcal{O}\Big\{&\frac{1}{\theta^2D^2}\,\|\overset{\cdot\,(k-2)}{t_1}\|_k + \frac{1}{\theta^2D^2}\,\|[(t_1t_2+t_1^3)F(t_1,t_2)]^{'(k-2)}\|_k \\
&+ \frac{\theta D}{h}\,\|\zeta_k[(t_1t_2+t_1^3)F(t_1,t_2)]^{'\cdot(k-1)}\|_k \\
&+ \|\zeta_k[(t_1t_2+t_1^3)F(t_1,t_2)]^{'(k)}\|_k\Big\} + \mathcal{O}\Big(\frac{h}{\theta D}\,B_{k1}\Big) \\
A_{k2} = \mathcal{O}\Big\{&\frac{1}{\theta^2D^2}\,\|\overset{\cdot\,(k-2)}{t_2}\|_k + \frac{1}{\theta^2D^2}\,\|[(t_1^2+t_2^2)F(t_1,t_2)]^{'(k-2)}\|_k \\
&+ \frac{\theta D}{h}\,\|\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]^{'\cdot(k-1)}\|_k \\
&+ \|\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]^{'(k)}\|_k\Big\} + \mathcal{O}\Big(\frac{h}{\theta D}\,B_{k2}\Big)
\end{aligned}\right.$$

which is valid for $k = 2,3,\ldots$.

From (3.3), line 10, we obtain

$$\begin{aligned}
\|\zeta_k\{\overset{\cdot\,(k-1)}{t}_{\alpha\beta\omega,3}\}\|_k &= \mathcal{O}\Big[\frac{1}{\theta hD}\,\|\{\overset{\cdot\,(k-2)}{t}_{\alpha\beta\omega}\}\|_k + \frac{\theta D}{h}\,\|\zeta_k\{\overset{\cdot\,(k)}{t}_{\alpha\beta\omega}\}\|_k \\
&\qquad + \frac{h}{\theta D}\,\|\zeta_k\{\overset{\cdot\,(k-2)}{t}_{\alpha\beta\omega,33}\}\|_k\Big] \\
&= \mathcal{O}\Big(\frac{1}{\theta hD}\,\|\overset{\cdot\,(k-2)}{t_\omega}\|_k + \frac{\theta D}{h}\,A_{k\omega}\Big) .
\end{aligned}$$

From this, (3.4), and (2.1), we have

$$(3.7)\left\{\begin{aligned}
B_{k1} &= \mathcal{O}\Big\{\frac{1}{\theta hD}\,\|\overset{\cdot\,(k-2)}{t_1}\|_k + \frac{\theta D}{h}\,A_{k1} + \|\zeta_k[(t_1t_2+t_1^3)F(t_1,t_2)]^{'(k)}\|_k\Big\} \\
B_{k2} &= \mathcal{O}\Big\{\frac{1}{\theta hD}\,\|\overset{\cdot\,(k-2)}{t_2}\|_k + \frac{\theta D}{h}\,A_{k2} + \|\zeta_k[(t_1^2+t_2^2)F(t_1,t_2)]^{'(k)}\|_k\Big\};
\end{aligned}\right.$$

which is valid for $k = 2,3,\ldots$.

From the last two equations in (1.4) we have

$$|t_\omega| = \mathcal{O}(\mu\,\bar\varepsilon_\omega) \tag{3.8}$$

where we recall that $\bar\varepsilon_2 = \varepsilon_2 + \frac{D^2}{h^2}\varepsilon_1^2$ and $\bar\varepsilon_1 = \varepsilon_1$.

From (3.3) and (3.8)

$$\begin{aligned} \|\zeta_2\dot t_\omega^2\|_2 &= \mathcal{O}\left(\mu\,\bar\varepsilon_\omega A_{2\omega} + \frac{\mu^2\sqrt{h}}{\theta\, D}\,\bar\varepsilon_\omega^2\right) \\ \|\zeta_2 t_\omega'^2\|_2 &= \mathcal{O}\left(\mu\,\bar\varepsilon_\omega B_{2\omega} + \frac{\mu^2\theta D}{h\,\sqrt{h}}\,\bar\varepsilon_\omega^2\right) . \end{aligned} \tag{3.9}$$

Differentiating we obtain

$$\left\{\begin{aligned}
[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot} &= (\dot t_1t_2+t_1\dot t_2+t_1^2\dot t_1)F(t_1,t_2) \\
[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\,\prime} &= (t_1''t_2+t_1t_2''+t_1^2t_1''+\dot t_1t_2'+t_1'\dot t_2+t_1t_1'^2 \\
&\qquad + t_2t_1'^2+t_1t_2'^2)F(t_1,t_2) \\
[(t_1t_2+t_1^3)F(t_1,t_2)]'' &= (t_1''t_2+t_1t_2''+t_1^2t_1''+t_1't_2'+t_1t_1'^2+t_2t_1'^2 \\
&\qquad + t_1t_2'^2)F(t_1,t_2) \\
[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot} &= (t_1\dot t_1+t_2\dot t_2+t_2^2\dot t_1+t_1^2\dot t_2)F(t_1,t_2) \\
[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot\,\prime} &= (t_1t_1''+t_2t_2''+t_2^2t_1''+t_1^2t_2''+\dot t_1t_1'+\dot t_2t_2'+t_2t_1'^2 \\
&\qquad +t_1t_1't_2'+t_1t_1'^2+t_2t_1't_2'+t_2t_2'^2+t_1^2t_2'^2)F(t_1,t_2) \\
[(t_1^2+t_2^2)F(t_1,t_2)]'' &= (t_1t_1''+t_2t_2''+t_2^2t_1''+t_1^2t_2''+t_1'^2+t_2'^2+ \\
&\qquad + t_1't_2')F(t_1,t_2) .
\end{aligned}\right. \tag{3.10}$$

From (3.10) and $\bar{\varepsilon}_\omega \le 2h^2/D^2 \le 2\theta^2$ and $\theta < 1$ we obtain

$$|[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\,\prime}| = \mathcal{O}\Big(\frac{h^2}{D^2}|t_1''|+\varepsilon_1|t_2''|+\frac{\theta D}{\mu h}\frac{\bar{\varepsilon}_2}{\varepsilon_1}|\dot{t}_1|^2 + \frac{h}{\mu\theta D}\frac{\varepsilon_1}{\bar{\varepsilon}_2}|t_2'|^2 + \frac{h}{\mu\theta D}\frac{\bar{\varepsilon}_2}{\varepsilon_1}|t_1'|^2 + \frac{\theta D}{\mu h}\frac{\varepsilon_1}{\bar{\varepsilon}_2}|\dot{t}_2|^2 + \frac{h^2}{\mu D^2}|t_1'|^2 + \frac{\varepsilon_1}{\mu}|t_2'|^2\Big)$$

$$|[(t_1t_2+t_1^3)F(t_1,t_2)]''| = \mathcal{O}\Big(\frac{h^2}{D^2}|t_1''|+\varepsilon_1|t_2''|+\frac{\bar{\varepsilon}_2}{\mu\bar{\varepsilon}_1}|t_1'|^2 + \frac{\varepsilon_1}{\mu\bar{\varepsilon}_2}|t_2'|^2+\frac{h^2}{\mu D^2}|t_1'|^2+\frac{\varepsilon_1}{\mu}|t_2'|^2\Big)$$

$$|[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot\,\prime}| = \mathcal{O}\Big(\frac{h^2}{D^2}|t_2''|+(\varepsilon_1+\bar{\varepsilon}_2)|t_1''|+\frac{\theta D}{\mu h}|\dot{t}_1|^2+\frac{h}{\mu\theta D}|t_1'|^2+\frac{\theta D}{\mu h}|\dot{t}_2|^2+\frac{h}{\mu\theta D}|t_2'|^2\Big)$$

$$|[(t_1^2+t_2^2)F(t_1,t_2)]''| = \mathcal{O}\Big(\frac{h^2}{D^2}|t_2''|+(\varepsilon_1+\bar{\varepsilon}_2)|t_1''|+\frac{|t_1'|^2+|t_2'|^2}{\mu}\Big)$$

From the above and (3.9)

$$\|\zeta_2[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\,\prime}\|_2 = \mathcal{O}\Big(\frac{h^2}{D^2}B_{21}+\varepsilon_1B_{22}+\frac{\theta h}{D}A_{21}+\frac{\theta D}{h}\varepsilon_1A_{22}+\frac{\mu h\sqrt{h}}{D^2}\varepsilon_1\Big)$$

$$\|\zeta_2[(t_1t_2+t_1^3)F(t_1,t_2)]''\|_2 = \mathcal{O}\Big(\frac{h^2}{D^2}B_{21}+\varepsilon_1B_{22}+\frac{\mu\sqrt{h}}{D}\varepsilon_1\Big)$$

$$\|\zeta_2[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot\,\prime}\|_2 = \mathcal{O}\Big(\frac{h^2}{D^2}B_{22}+(\varepsilon_1+\bar{\varepsilon}_2)B_{21}+\frac{\theta D}{h}\varepsilon_1A_{21}+\frac{\theta h}{D}A_{22}+\frac{\mu h\sqrt{h}}{D^2}\bar{\varepsilon}_2\Big)$$

$$\|\zeta_2[(t_1^2+t_2^2)F(t_1,t_2)]''\|_2 = \mathcal{O}\Big[\frac{h^2}{D^2}B_{22}+(\varepsilon_1+\bar{\varepsilon}_2)B_{21}+\frac{\mu\sqrt{h}}{D}\bar{\varepsilon}_2\Big].$$

Substituting into (3.6) and (3.7)

$$A_{21} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D}\,\varepsilon_1 + \frac{\theta h}{D}\,B_{21} + \frac{\theta D}{h}\,\varepsilon_1 B_{22} + \theta^2 A_{21} + \frac{\theta^2 D^2}{h^2}\,\varepsilon_1 A_{22}\right) + \mathcal{O}\left(\frac{h}{\theta D}\,B_{21}\right)$$

$$A_{22} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D}\,\bar{\varepsilon}_2 + \frac{\theta h}{D}\,B_{22} + \frac{\theta D}{h}(\varepsilon_1+\bar{\varepsilon}_2)B_{21} + \frac{\theta^2 D^2}{h^2}\,\varepsilon_1 A_{21} + \theta^2 A_{22}\right) + \mathcal{O}\left(\frac{h}{\theta D}\,B_{22}\right)$$

$$B_{21} = \mathcal{O}\left(\frac{\mu\varepsilon_1}{\sqrt{h}} + \frac{\theta D}{h}\,A_{21} + \theta^2 B_{21} + \varepsilon_1 B_{22}\right)$$

$$B_{22} = \mathcal{O}\left[\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}} + \frac{\theta D}{h}A_{22} + \theta^2 B_{22} + (\varepsilon_1+\bar{\varepsilon}_2)B_{21}\right] .$$

The last of these has the form

$$B_{22} \leq K\left[\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}} + \frac{\theta D}{h}\,A_{22} + \theta^2 B_{22} + (\varepsilon_1+\bar{\varepsilon}_2)B_{21}\right]$$

where K can be chosen to be independent of θ. Hence choosing θ so that $K\theta^2 \leq 1/2$, we get

$$B_{22} \leq 2K\left[\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}} + \frac{\theta D}{h}\,A_{22} + (\varepsilon_1+\bar{\varepsilon}_2)B_{21}\right] .$$

Substituting this into the above

$$A_{21} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D}\,\varepsilon_1 + \frac{\theta h}{D}\,B_{21} + \theta^2 A_{21} + \frac{\theta^2 D^2}{h^2}\,\varepsilon_1 A_{22}\right) + \mathcal{O}\left(\frac{h}{\theta D}\,B_{21}\right)$$

$$A_{22} = \mathcal{O}\left[\frac{\mu\sqrt{h}}{\theta D}\,\bar{\varepsilon}_2 + \frac{\theta D}{h}\,(\varepsilon_1+\bar{\varepsilon}_2)B_{21} + \frac{\theta^2 D^2}{h^2}\,\varepsilon_1 A_{21} + \theta^2 A_{22}\right] + \mathcal{O}(A_{22})$$

$$B_{21} = \mathcal{O}\left[\frac{\mu\varepsilon_1}{\sqrt{h}} + \frac{\theta D}{h}\,A_{21} + \theta^2 B_{21} + \frac{\theta D}{h}\,\varepsilon_1 A_{22}\right]$$

$$B_{22} = \mathcal{O}\left[\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}} + \frac{\theta D}{h}\,A_{22} + (\varepsilon_1+\bar{\varepsilon}_2)B_{21}\right] .$$

Similarly we can eliminate B_{21} from the right-hand sides if θ is sufficiently small.

$$B_{21} = \mathcal{O}\left(\frac{\mu\varepsilon_1}{\sqrt{h}} + \frac{\theta D}{h} A_{21} + \frac{\theta D}{h} \varepsilon_1 A_{22}\right)$$

$$A_{21} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D} \varepsilon_1 + \theta^2 A_{21} + \frac{\theta^2 D^2}{h^2} \varepsilon_1 A_{22}\right) + o(A_{21})$$

$$A_{22} = \mathcal{O}\left[\frac{\mu\sqrt{h}}{\theta D} \bar{\varepsilon}_2 + \frac{\theta^2 D^2}{h^2} (\varepsilon_1+\bar{\varepsilon}_2)A_{21} + \theta^2 A_{22}\right] + o(A_{22})$$

$$B_{22} = \mathcal{O}\left[\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}} + \frac{\theta D}{h} (\varepsilon_1+\bar{\varepsilon}_2)A_{21} + \frac{\theta D}{h} A_{22}\right] .$$

The second of these has the form

$$A_{21} \leq K(\delta) \left[\frac{\mu\sqrt{h}}{\theta D} \varepsilon_1 + \theta^2 A_{21} + \frac{\theta^2 D^2}{h^2} \varepsilon_1 A_{22}\right] + \delta A_{21}$$

where $K(\delta)$ is a positive function which can be chosen to be independent of θ. Thus

$$A_{21} \leq 2K(\tfrac{1}{2}) \left[\frac{\mu\sqrt{h}}{\theta D} \varepsilon_1 + \theta^2 A_{21} + \frac{\theta^2 D^2}{h^2} \varepsilon_1 A_{22}\right] .$$

We choose θ so that $2K(\frac{1}{2})\theta^2 \leq \frac{1}{2}$. Then

$$A_{21} \leq 4K(\tfrac{1}{2}) \left[\frac{\mu\sqrt{h}}{\theta D} \varepsilon_1 + \frac{\theta^2 D^2}{h^2} \varepsilon_1 A_{22}\right] .$$

Substituting this into the above we have

$$A_{21} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D} \varepsilon_1 + \frac{\theta^2 D^2}{h^2} \varepsilon_1 A_{22}\right)$$

$$A_{22} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D} \bar{\varepsilon}_2 + \theta^2 A_{22}\right) + o(A_{22})$$

$$B_{21} = \mathcal{O}\left(\frac{\mu\varepsilon_1}{\sqrt{h}} + \frac{\theta^3 D^3}{h^3} \varepsilon_1 A_{22}\right)$$

$$B_{22} = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}} + \frac{\theta D}{h} A_{22}\right) .$$

We can eliminate A_{22} from the right-hand side by choosing θ sufficiently small in the same way as A_{21} was eliminated. This gives

$$A_{22} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D}\,\bar{\varepsilon}_2\right) , \quad A_{21} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta D}\,\varepsilon_1\right) , \quad B_{21} = \mathcal{O}\left(\frac{\mu\varepsilon_1}{\sqrt{h}}\right) , \quad B_{22} = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}}\right)$$

Thus we have proved (3.12) for k = 2.

$$(3.12) \qquad \left\{ \begin{aligned} A_{k\omega} &= \mathcal{O}\left(\frac{\mu\ \sqrt{h}}{(\theta D)^{k-1}}\,\bar{\varepsilon}_\omega\right) \\ B_{k\omega} &= \mathcal{O}\left(\frac{\mu\ \bar{\varepsilon}_\omega}{\sqrt{h}\ (\theta D)^{k-2}}\right) \end{aligned} \right.$$

for $k = 1,2,3,\ldots$.

Next we prove (3.12) for $k = 1$. From (3.3), (3.8), (3.12) for $k = 2$,

$$\|\zeta_1\dot{t}_\omega\|_1 = \mathcal{O}\left(\frac{1}{\theta D}\,\|t_\omega\|_1\right) + o(\theta D\|\zeta_2\ddot{t}_\omega\|_2) \qquad (\text{since } \zeta_1 \equiv \zeta_2)$$

$$= \mathcal{O}(\mu\sqrt{h}\ \bar{\varepsilon}_\omega) + o(\theta D A_{2\omega}) = \mathcal{O}(\mu\sqrt{h}\ \bar{\varepsilon}_\omega)$$

$$\|\zeta_1 t'_\omega\|_1 = \mathcal{O}\left(\frac{1}{h}\,\|t_\omega\|_1\right) + o(\theta D\|\zeta_2\ddot{t}_\omega\|_2 + h\|\zeta_2 t_{\omega,33}\|_2)$$

$$= \mathcal{O}\left(\frac{\mu\theta D}{\sqrt{h}}\,\bar{\varepsilon}_\omega\right) + o[(\theta D+h)A_{2\omega}] = \mathcal{O}\left(\frac{\mu\theta D}{\sqrt{h}}\,\bar{\varepsilon}_\omega\right) .$$

Also from (3.10) and the above

$$\|\zeta_1[(t_1t_2+t_1^3)F(t_1,t_2)]'\|_1 = \mathcal{O}(\bar{\varepsilon}_2\|\zeta_1t'_1\|_1 + \varepsilon_1\|\zeta_1t'_2\|_1) = \mathcal{O}\left(\frac{\mu\theta h\sqrt{h}}{D}\,\varepsilon_1\right)$$

$$\|\zeta_1[(t_1^2+t_2^2)F(t_1,t_2)]'\|_1 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2^2)\|\zeta_1t'_1\|_1 + \bar{\varepsilon}_2\|\zeta_1t'_2\|_1] = \mathcal{O}\left(\frac{\mu\theta h\sqrt{h}}{D}\,\bar{\varepsilon}_2\right)$$

Using the above results in (2.1), we obtain (3.12) for k = 1.

Next we prove (3.12) for k =3. For the present we let

$$m_\omega = \max \zeta_3|\dot{t}_\omega| , \quad M_\omega = \max \zeta_3|t'_\omega| .$$

From (3.10)

$$
(3.13)\left\{\begin{aligned}
&[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\cdot\prime} = [t_1^{\prime\prime\prime} t_2+t_1t_2^{\prime\prime\prime} +t_1^2t_1^{\prime\prime\prime} +t_1^{\prime\prime}\dot{t}_2+\dot{t}_1t_2^{\prime\prime}+t_1t_1^{\prime}t_1^{\prime\prime} \\
&\quad + \ddot{t}_1t_2^{\prime}+t_1^{\prime}\ddot{t}_2+t_2t_1^{\prime}t_1^{\prime\prime}+t_1t_2^{\prime}t_2^{\prime\prime}+t_1^{\prime 3}+t_1^{\prime 2}t_2^{\prime}+t_1^{\prime}t_2^{\prime 2}+t_1t_2^{\prime 3}]F(t_1,t_2) \\
&[(t_1t_2+t_1^3)F(t_1,t_2)]^{\prime\prime\prime} = (t_1^{\prime\prime\prime} t_2+t_1t_2^{\prime\prime\prime}+t_1^2t_1^{\prime\prime\prime} +t_1^{\prime\prime}t_2^{\prime}+t_1^{\prime}t_2^{\prime\prime}+t_1t_1^{\prime}t_1^{\prime\prime} \\
&\quad + t_2t_1^{\prime}t_1^{\prime\prime}+t_1t_2^{\prime}t_2^{\prime\prime}+t_1^{\prime 3}+t_1^{\prime 2}t_2^{\prime}+t_1^{\prime}t_2^{\prime 2}+t_1t_2^{\prime 3})F(t_1,t_2) \\
&[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot\cdot\prime} = (t_1t_1^{\prime\prime\prime}+t_2t_2^{\prime\prime\prime}+t_2^2t_1^{\prime\prime\prime} +t_1^2t_2^{\prime\prime\prime} +\dot{t}_1t_1^{\prime\prime}+\dot{t}_2t_2^{\prime\prime}+t_2t_2^{\prime}t_1^{\prime\prime} \\
&\quad + t_1t_1^{\prime}t_2^{\prime\prime}+\ddot{t}_1t_1^{\prime}+\ddot{t}_2t_2^{\prime}+\ddot{t}_2t_2^{\prime}+t_2t_1^{\prime}t_1^{\prime\prime}+t_1t_2^{\prime}t_1^{\prime\prime}+t_1t_1^{\prime}t_1^{\prime\prime}+t_2t_1^{\prime}t_2^{\prime\prime}+t_2t_2^{\prime}t_2^{\prime\prime} \\
&\quad + t_1^2t_2^{\prime}t_2^{\prime\prime}+t_1^{\prime 3}+t_1^{\prime 2}t_2^{\prime}+t_1^{\prime}t_2^{\prime 2}+t_2^{\prime 3})F(t_1,t_2) \\
&[(t_1^2+t_2^2)F(t_1,t_2)]^{\prime\prime\prime} = (t_1t_1^{\prime\prime\prime}+t_2t_2^{\prime\prime\prime} +t_2^2t_1^{\prime\prime\prime} +t_1^2t_2^{\prime\prime\prime}+t_1^{\prime}t_1^{\prime\prime}+t_2^{\prime}t_2^{\prime\prime} \\
&\quad + t_2^{\prime}t_1^{\prime\prime}+t_1^{\prime}t_2^{\prime\prime}+t_1^{\prime 3}+t_1^{\prime 2}t_2^{\prime}+t_1^{\prime}t_2^{\prime 2}+t_2^{\prime 3})F(t_1,t_2)
\end{aligned}\right.
$$

From (3.13) and the fact that $\zeta_3 = \mathcal{O}(\zeta_3\zeta_2)$

$$
\begin{aligned}
\zeta_3|[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\cdot\prime}| = \mathcal{O}\Big\{&\frac{h^2}{D^2}\,\zeta_3|t_1^{\prime\prime\prime}| + \varepsilon_1\zeta_3|t_2^{\prime\prime\prime}| + \frac{M_2}{\mu}\,\zeta_2|\ddot{t}_1| \\
&+ \frac{M_1}{\mu}\,\zeta_2|\ddot{t}_2| \\
&+ \frac{1}{\mu}\,[m_2+(\varepsilon_1+\bar{\varepsilon}_2)M_1]\zeta_2|t_1^{\prime\prime}| + \frac{1}{\mu}\,(m_1+\varepsilon_1M_2)\zeta_2|t_2^{\prime\prime}| \\
&+ \frac{1}{\mu^2}\,(M_1+M_2)\zeta_2|t_1^{\prime}|^2 + \frac{1}{\mu^2}\,(M_1+\varepsilon_1M_2)\zeta_2|t_2^{\prime}|^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\zeta_3|[(t_1t_2+t_1^3)F(t_1,t_2)]^{\prime\prime\prime}| = \mathcal{O}\Big\{&\frac{h^2}{D^2}\zeta_3|t_1^{\prime\prime\prime}| + \varepsilon_1\zeta_3|t_2^{\prime\prime\prime}| + [M_2+(\varepsilon_1+\bar{\varepsilon}_2)M_1]\zeta_2\frac{|t_1^{\prime\prime}|}{\mu} \\
&+ (M_1+\varepsilon_1M_2)\zeta_2\,\frac{|t_2^{\prime\prime}|}{\mu} + (M_1+M_2)\zeta_2\,\frac{|t_1^{\prime}|^2}{\mu^2} + (M_1+\varepsilon_1M_2)\zeta_2\,\frac{|t_2^{\prime}|^2}{\mu^2}\Big\}
\end{aligned}
$$

$$\zeta_3 \left| \left[(t_1^2+t_2^2) F(t_1,t_2) \right]^{\cdot\cdot\prime} \right| = \mathcal{O}\Big\{ (\varepsilon_1+\bar{\varepsilon}_2)\zeta_3 |t_1'''| + \frac{h^2}{D^2}\zeta_3 |t_2'''| + M_1\zeta_2 \frac{|\ddot{t}_1|}{\mu} + M_2\zeta_2 \frac{|\ddot{t}_2|}{\mu}$$

$$+ [m_1+(\varepsilon_1+\bar{\varepsilon}_2)(M_1+M_2)]\zeta_2 \frac{|t_1''|}{\mu} + [m_2+(\varepsilon_1+\bar{\varepsilon}_2)(M_1+M_2)]\zeta_2 \frac{|t_2''|}{\mu}$$

$$+ (M_1+M_2)\zeta_2 \frac{|t_1'|^2}{\mu^2} + (M_1+M_2)\zeta_2 \frac{|t_2'|^2}{\mu^2} \Big\}$$

$$\zeta_3 \left| \left[(t_1^2+t_2^2) F(t_1,t_2) \right]''' \right| = \mathcal{O}\Big\{ (\varepsilon_1+\bar{\varepsilon}_2)\zeta_3 |t_1'''| + \frac{h^2}{D^2}\zeta_3 |t_2'''|$$

$$+ (M_1+M_2)\zeta_2 \left(\frac{|t_1''|}{\mu} + \frac{|t_2''|}{\mu} + \frac{|t_1'|^2}{\mu^2} + \frac{|t_2'|^2}{\mu^2} \right) \Big\}$$

Using the above, (3.12) for k = 2, and (3.9)

$$\| \zeta_3 [(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\cdot\prime} \|_3 = \mathcal{O}\Big[\frac{h^2}{D^2} B_{31} + \varepsilon_1 B_{32} + \frac{\bar{\varepsilon}_2}{\sqrt{h}} m_1 + \frac{\varepsilon_1}{\sqrt{h}} m_2$$

$$+ \frac{\sqrt{h}}{\theta D} (\varepsilon_1+\bar{\varepsilon}_2) M_1 + \frac{\sqrt{h}}{\theta D} \varepsilon_1 M_2 \Big]$$

$$\| \zeta_3 [(t_1t_2+t_1^3)F(t_1,t_2)]''' \|_3 = \mathcal{O}\Big(\frac{h^2}{D^2} B_{31} + \varepsilon_1 B_{32} + M_1 \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}} + M_2 \frac{\varepsilon_1}{\sqrt{h}} \Big)$$

$$\| \zeta_3 [(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot\cdot\prime} \|_3 = \mathcal{O}\Big[(\varepsilon_1+\bar{\varepsilon}_2) B_{31} + \frac{h^2}{D^2} B_{32} + \frac{m_1\varepsilon_1}{\sqrt{h}} + \frac{m_2\bar{\varepsilon}_2}{\sqrt{h}}$$

$$+ (M_1+M_2) \frac{\sqrt{h}}{\theta D} (\varepsilon_1+\bar{\varepsilon}_2) \Big]$$

$$\| \zeta_3 [(t_1^2+t_2^2)F(t_1,t_2)]''' \|_3 = \mathcal{O}\Big[(\varepsilon_1+\bar{\varepsilon}_2) B_{31} + \frac{h^2}{D^2} B_{32} + \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}} (M_1+M_2) \Big].$$

Also

$$\| \dot{t}_\omega \|_3 = \mathcal{O}(\| \zeta_1 \dot{t}_\omega \|_1) = \mathcal{O}(A_{1\omega}) = \mathcal{O}(\mu\sqrt{h}\,\bar{\varepsilon}_\omega) ,$$

and from (3.10)

$$\|[(t_1t_2+t_1^3)F(t_1,t_2)]'\|_3 = \mathcal{O}(\frac{h^2}{D^2} B_{11} + \varepsilon_1 B_{12}) = \mathcal{O}(\frac{\mu\theta h\sqrt{h}}{D} \varepsilon_1)$$

$$\|[(t_1^2+t_2^2)F(t_1,t_2)]'\|_3 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)B_{11} + \frac{h^2}{D^2} B_{12}] = \mathcal{O}(\frac{\mu\theta h\sqrt{h}}{D} \bar{\varepsilon}_2) .$$

Using the above results, (3.6), and (3.7)

$$A_{31} = \mathcal{O}[\frac{\mu\sqrt{h}\varepsilon_1}{\theta^2D^2} + \frac{\theta h}{D} B_{31} + \frac{\theta D}{h}\varepsilon_1 B_{32} + \frac{\theta D}{h\sqrt{h}}(\bar{\varepsilon}_2 m_1 + \varepsilon_1 m_2) + \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}} M_1 + \frac{\varepsilon_1}{\sqrt{h}} M_2] + \mathcal{O}(\frac{h}{\theta D} B_{31})$$

$$A_{32} = \mathcal{O}[\frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{\theta^2D^2} + \frac{\theta D}{h}(\varepsilon_1+\bar{\varepsilon}_2)B_{31} + \frac{\theta h}{D} B_{32} + \frac{\theta D}{h\sqrt{h}}(m_1\varepsilon_1+m_2\bar{\varepsilon}_2) + \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}}(M_1+M_2)] + \mathcal{O}(\frac{h}{\theta D} B_{32})$$

$$B_{31} = \mathcal{O}(\frac{\mu\varepsilon_1}{\theta\sqrt{h}\,D} + \frac{\theta D}{h} A_{31} + \theta^2 B_{31} + \varepsilon_1 B_{32} + M_1\frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}} + M_2\frac{\varepsilon_1}{\sqrt{h}})$$

$$B_{32} = \mathcal{O}[\frac{\mu\bar{\varepsilon}_2}{\theta\sqrt{h}\,D} + \frac{\theta D}{h} A_{32} + (\varepsilon_1+\bar{\varepsilon}_2)B_{31} + \theta^2 B_{32} + \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}}(M_1+M_2)] .$$

$A_{3\omega}$ and $B_{3\omega}$ can be eliminated from the right-hand sides of the above equations when θ is sufficiently small, just as when $k = 2$, to obtain

$$(3.14)\begin{cases} A_{31} = \mathcal{O}[\frac{\mu\sqrt{h}\,\varepsilon_1}{\theta^2D^2} + \frac{\theta D}{h\sqrt{h}}(\bar{\varepsilon}_2 m_1+\varepsilon_1 m_2) + \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}} M_1 + \frac{\varepsilon_1}{\sqrt{h}} M_2] \\ A_{32} = \mathcal{O}\left\{\frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{\theta^2D^2} + \frac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}}[\frac{\theta D}{h}(m_1+m_2) + M_1 + M_2]\right\} \\ B_{31} = \mathcal{O}\left\{\frac{\mu\varepsilon_1}{\theta\sqrt{h}\,D} + \frac{\theta D}{h\sqrt{h}}[M_1(\varepsilon_1+\bar{\varepsilon}_2) + M_2\varepsilon_1] + \frac{\theta^2D^2}{h^2\sqrt{h}}(m_1\bar{\varepsilon}_2 + m_2\varepsilon_1)\right\} \\ B_{32} = \mathcal{O}\left\{\frac{\mu\bar{\varepsilon}_2}{\theta\sqrt{h}\,D} + \frac{\theta D}{h\sqrt{h}}(\varepsilon_1+\bar{\varepsilon}_2)[M_1 + M_2 + \frac{\theta D}{h}(m_1+m_2)]\right\} . \end{cases}$$

From the last line of (3.3)

$$\frac{\sqrt{h}}{\theta D}\,\zeta_3|\dot{t}_\omega| = \mathcal{O}\left(A_{3\omega} + \frac{1}{\theta^2 D^2}\,\|\dot{t}_\omega\|_3 + \frac{h^2}{\theta^2 D^2}\,B_{3\omega}\right) .$$

Since $\|\dot{t}_\omega\|_3 = \mathcal{O}(\|\zeta_1\dot{t}_\omega\|_3) = \mathcal{O}(\|\zeta_1\dot{t}_\omega\|_1) = \mathcal{O}(A_{1\omega}) = \mathcal{O}(\mu\sqrt{h}\,\bar{\varepsilon}_\omega)$,
we have

$$\frac{\sqrt{h}}{\theta D}\,\zeta_3|\dot{t}_\omega| = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta^2 D^2}\,\bar{\varepsilon}_\omega + A_{3\omega} + \frac{h^2}{\theta^2 D^2}\,B_{3\omega}\right)$$

so that

$$m_\omega = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta D} + \frac{\theta D}{\sqrt{h}}\,A_{3\omega} + \frac{h\sqrt{h}}{\theta D}\,B_{3\omega}\right) .$$

Similarly,

$$\frac{\sqrt{h}}{\theta D}\,\zeta_3|t'_\omega| = \mathcal{O}\left(B_{3\omega} + \frac{1}{\theta^2 D^2}\,\|t'_\omega\|_3\right) = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\sqrt{h}\;\theta D} + B_{3\omega}\right)$$

$$M_\omega = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h} + \frac{\theta D}{\sqrt{h}}\,B_{3\omega}\right) .$$

Using (3.14) with the above, we obtain

$$m_1 = \mathcal{O}\left\{\frac{\mu\varepsilon_1}{\theta D} + \theta^2 m_1 + \frac{\theta^2 D^2}{h^2}\,\varepsilon_1 m_2 + \frac{\theta D}{h}\,[(\varepsilon_1+\bar{\varepsilon}_2)M_1 + \varepsilon_1 M_2]\right\}$$

$$m_2 = \mathcal{O}\,[\frac{\mu\bar{\varepsilon}_2}{\theta D} + \theta^2 m_2 + \frac{\theta D}{h}\,(\varepsilon_1+\bar{\varepsilon}_2)\;(\frac{\theta D}{h}\,m_1 + M_1 + M_2)]$$

$$M_1 = \mathcal{O}\left\{\frac{\mu\varepsilon_1}{h} + \theta^2 M_1 + \frac{\theta^2 D^2}{h^2}\,[M_2\varepsilon_1 + \frac{\theta D}{h}\,(m_1\bar{\varepsilon}_2 + m_2\varepsilon_1)]\right\}$$

$$M_2 = \mathcal{O}\left\{\frac{\mu\bar{\varepsilon}_2}{h} + \theta^2 M_2 + \frac{\theta^2 D^2}{h^2}\,(\varepsilon_1+\bar{\varepsilon}_2)\;[M_1 + \frac{\theta D}{h}\,(m_1+m_2)]\right\} .$$

When θ is sufficiently small, we can eliminate m_ω, M_ω from the right-hand sides to obtain

$$m_\omega = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta D}\right) , \qquad M_\omega = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h}\right) .$$

Substituting the above into (3.14), we obtain (3.12) for k = 3. From the above we also obtain

$$\zeta_3|\dot{t}_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta D}\right), \qquad \zeta_3|t'_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h}\right).$$

Since ζ_3 has a positive minimum in the set where $\zeta_4 \neq 0$, we have

$$|\dot{t}_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta D}\right), \qquad |t'_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h}\right)$$

in the set where $|x_3| \leq h$ and $\zeta_4 \neq 0$. From this, (2.1), and (3.10), we obtain (3.15) for k = 1.

$$(3.15) \qquad |a_{k\omega}| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{(\theta D)^k}\right), \qquad |b_{k\omega}| = \mathcal{O}\left[\frac{\mu\bar{\varepsilon}_\omega}{h(\theta D)^{k-1}}\right], \quad k=1,2,\ldots,$$

valid where $|x_3| \leq h$ and $\zeta_{k+3} \neq 0$.

Next we change the meanings of m_ω and M_ω to

$$m_\omega = \max \zeta_4|\ddot{t}_\omega|, \qquad M_\omega = \max \zeta_4|t''_\omega|.$$

Differentiating (3.13) and using the estimates for t_ω, $\dot{t}_\omega$, t'_ω, we have the following where $|x_3| \leq h$ and $\zeta_4 \neq 0$.

$$\zeta_4|[(t_1t_2+t_1^3)F(t_1,t_2)]^{\dddot{}\,\prime}| = \mathcal{O}\Big[\frac{h^2}{D^2}\zeta_4|t_1''''| + \varepsilon_1\zeta_4|t_2''''| + \frac{h}{D^2}\zeta_3|\dddot{t}_1| + \frac{h^2}{\theta D^3}\zeta_3|t_1'''|$$
$$+ \frac{\varepsilon_1}{h}\zeta_3|\dddot{t}_2| + \frac{\varepsilon_1}{\theta D}\zeta_3|t_2'''| + \left(\frac{m_2}{\mu} + \frac{h^2}{D^2}\frac{M_1}{\mu} + \frac{h^2}{D^4}\right)\zeta_2|t_1''|$$
$$+ \left(\frac{m_1}{\mu} + \varepsilon_1\frac{M_2}{\mu} + \frac{\varepsilon_1}{D^2}\right)\zeta_2|t_2''| + \frac{\mu h^2\varepsilon_1}{D^6}\Big]$$

$$\zeta_4|[(t_1t_2+t_1^3)F(t_1,t_2)]''''| = \mathcal{O}\Big[\frac{h^2}{D^2}\zeta_4|t_1''''| + \varepsilon_1\zeta_4|t_2''''| + \frac{h}{D^2}\zeta_3|t_1'''|$$
$$+ \frac{\varepsilon_1}{h}\zeta_3|t_2'''| + \left(\frac{M_1+M_2}{\mu} + \frac{h^2}{D^4}\right)\zeta_2|t_1''| + \left(\frac{M_2}{\mu} + \frac{1}{D^2}\right)\varepsilon_1\zeta_2|t_2''| + \frac{\mu h^2\varepsilon_1}{D^6}\Big]$$

$$\zeta_4|[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdots\prime}| = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)\zeta_4|t_1''''| + \frac{h^2}{D^2}\zeta_4|t_2''''|$$

$$+ \frac{\varepsilon_1+\bar{\varepsilon}_2}{\theta D}\zeta_3|t_1'''| + \frac{h^2}{\theta D^3}\zeta_3|t_2'''| + \frac{\varepsilon_1}{h}\zeta_3|\dddot{t}_1| + \frac{h}{D^2}\zeta_3|\dddot{t}_2|$$

$$+ (\frac{m_1}{\mu} + h^2\frac{M_1+M_2}{\mu} + \frac{\bar{\varepsilon}_2}{D^2})\zeta_2|t_1''| + (\frac{m_2}{\mu} + \frac{h^2}{D^2}\frac{M_2}{\mu} + \frac{h^2}{D^4})\zeta_2|t_2''| + \frac{\mu h^2}{D^6}\bar{\varepsilon}_2]$$

$$\zeta_4|[(t_1^2+t_2^2)F(t_1,t_2)]''''| = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)\zeta_4|t_1''''| + \frac{h^2}{D^2}\zeta_4|t_2''''|$$

$$+ \frac{\varepsilon_1+\bar{\varepsilon}_2}{h}\zeta_3|t_1'''| + \frac{h}{D^2}\zeta_3|t_2'''| + (\frac{M_1+M_2}{\mu} + \frac{\bar{\varepsilon}_2}{D^2})\zeta_2|t_1''|$$

$$+ (\frac{M_2}{\mu} + \frac{h^2}{D^4})\zeta_2|t_2''| + \frac{\mu\bar{\varepsilon}_2 h^2}{D^6}] .$$

Hence

$$\|\zeta_4[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdots\prime}\|_4 = \mathcal{O}[\frac{h^2}{D^2}B_{41} + \varepsilon_1 B_{42} + \frac{\mu h\sqrt{h}}{\theta^2 D^4}\varepsilon_1 + \frac{\varepsilon_1}{\sqrt{h}}m_2$$

$$+ \frac{h\sqrt{h}}{D^2}m_1 + \frac{h^3\sqrt{h}}{D^4}M_1 + \frac{h\sqrt{h}}{D^2}\varepsilon_1 M_2]$$

$$\|\zeta_4[(t_1t_2+t_1^3)F(t_1,t_2)]''''\|_4 = \mathcal{O}[\frac{h^2}{D^2}B_{41} + \varepsilon_1 B_{42} + \frac{\mu\sqrt{h}}{\theta D^3}\varepsilon_1 + \frac{h\sqrt{h}}{D^2}M_1 + \frac{\varepsilon_1}{\sqrt{h}}M_2]$$

$$\|\zeta_4[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdots\prime}\|_4 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)B_{41} + \frac{h^2}{D^2}B_{42} + \frac{\mu h\sqrt{h}}{\theta^2 D^4}\bar{\varepsilon}_2 + \frac{\varepsilon_1}{\sqrt{h}}m_1$$

$$+ \frac{h\sqrt{h}}{D^2}m_2 + \frac{h\sqrt{h}}{D^2}\varepsilon_1 M_1 + \frac{h^3\sqrt{h}}{D^4}M_2]$$

$$\|\zeta_4[(t_1^2+t_2^2)F(t_1,t_2)]''''\|_4 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)B_{41} + \frac{h^2}{D^2}B_{42} + \frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{\theta D^3} + \frac{\varepsilon_1}{\sqrt{h}}M_1$$

$$+ \frac{h\sqrt{h}}{D^2}M_2] .$$

Also from (3.12) and (3.10)

$$\|\ddot{t}_\omega\|_4 = \mathcal{O}(\|\zeta_2\ddot{t}_\omega\|_2) = \mathcal{O}(A_{2\omega}) = \mathcal{O}(\frac{\mu\sqrt{h}\,\bar{\varepsilon}_\omega}{\theta D})$$

$$\|[t_1t_2+t_1^3)F(t_1,t_2)]''\|_4 = \mathcal{O}(\frac{h^2}{D^2}B_{21}+\varepsilon_1B_{22}+\frac{\mu\theta\sqrt{h}\,\varepsilon_1}{D}) = \mathcal{O}(\frac{\mu\sqrt{h}\,\varepsilon_1}{D})$$

$$\|[(t_1^2+t_2^2)F(t_1,t_2)]''\|_4 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)B_{21}+\frac{h^2}{D^2}B_{22}+\frac{\mu\theta\sqrt{h}\,\bar{\varepsilon}_2}{D}] = \mathcal{O}(\frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{D})$$

Using these in (3.6) and (3.7), we have

$$A_{41} = \mathcal{O}[\frac{\mu\sqrt{h}\,\varepsilon_1}{\theta^3D^3} + \frac{\theta h}{D}B_{41} + \frac{\theta D}{h}\varepsilon_1B_{42} + \frac{\theta D}{h\sqrt{h}}\varepsilon_1m_2 + \frac{\theta\sqrt{h}}{D}m_1$$
$$+ \frac{h\sqrt{h}}{D^2}M_1 + \frac{\varepsilon_1}{\sqrt{h}}M_2] + o(\frac{h}{\theta D}B_{41})$$

$$A_{42} = \mathcal{O}[\frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{\theta^3D^3} + \frac{\theta D}{h}(\varepsilon_1+\bar{\varepsilon}_2)B_{41} + \frac{\theta h}{D}B_{42} + \frac{\theta D}{h\sqrt{h}}\varepsilon_1m_1 + \frac{\theta\sqrt{h}}{D}m_2$$
$$+ \frac{\varepsilon_1}{\sqrt{h}}M_1 + \frac{h\sqrt{h}}{D^2}M_2] + o(\frac{h}{\theta D}B_{42})$$

$$B_{41} = \mathcal{O}[\frac{\mu\varepsilon_1}{\sqrt{h}\,\theta^2D^2} + \theta^2B_{41} + \varepsilon_1B_{42} + \frac{\theta D}{h}A_{41} + \frac{h\sqrt{h}}{D^2}M_1 + \frac{\varepsilon_1}{\sqrt{h}}M_2]$$

$$B_{42} = \mathcal{O}[\frac{\mu\bar{\varepsilon}_2}{\sqrt{h}\,\theta^2D^2} + \frac{\theta D}{h}A_{42} + (\varepsilon_1+\bar{\varepsilon}_2)B_{41} + \theta^2B_{42} + \frac{\varepsilon_1}{\sqrt{h}}M_1 + \frac{h\sqrt{h}}{D^2}M_2]$$

For θ sufficiently small, we can eliminate $A_{4\omega}$, $B_{4\omega}$ from the right-hand sides to obtain

$$(3.16)\begin{cases} A_{41} = \mathcal{O}[\dfrac{\mu\sqrt{h}\ \varepsilon_1}{\theta^3 D^3} + \dfrac{\theta D}{h\sqrt{h}}\varepsilon_1 m_2 + \dfrac{\theta\sqrt{h}}{D} m_1 + \dfrac{h\sqrt{h}}{D^2} M_1 + \dfrac{\varepsilon_1}{\sqrt{h}} M_2] \\ A_{42} = \mathcal{O}[\dfrac{\mu\sqrt{h}\ \bar{\varepsilon}_2}{\theta^3 D^3} + \dfrac{\theta D}{h\sqrt{h}}(\varepsilon_1+\bar{\varepsilon}_2) m_1 + \dfrac{\theta\sqrt{h}}{D} m_2 + \dfrac{\varepsilon_1+\bar{\varepsilon}_2}{\sqrt{h}} M_1 + \dfrac{h\sqrt{h}}{D^2} M_2] \\ B_{41} = \mathcal{O}[\dfrac{\mu\ \varepsilon_1}{\sqrt{h}\ \theta^2 D^2} + \dfrac{\theta\sqrt{h}}{D} M_1 + \dfrac{\theta D}{h\sqrt{h}}\varepsilon_1 M_2 + \dfrac{\theta^2}{\sqrt{h}} m_1 + \dfrac{\theta^2 D^2}{h^2\ \sqrt{h}}\varepsilon_1 m_2] \\ B_{42} = \mathcal{O}[\dfrac{\mu\ \bar{\varepsilon}_2}{\sqrt{h}\ \theta^2 D^2} + \dfrac{\theta D}{h\sqrt{h}}(\varepsilon_1+\bar{\varepsilon}_2) M_1 + \dfrac{\theta\sqrt{h}}{D} M_2 + \dfrac{\theta^2 D^2}{h^2\ \sqrt{h}}(\varepsilon_1+\bar{\varepsilon}_2) m_1 + \dfrac{\theta^2}{\sqrt{h}} m_2] \end{cases}$$

As before, the last line of (3.3) gives

$$\frac{\sqrt{h}}{\theta D}\zeta_4|\ddot{t}_\omega| = \mathcal{O}(A_{4\omega} + \frac{1}{\theta^2 D^2}\|\ddot{t}_\omega\|_4 + \frac{h^2}{\theta^2 D^2} B_{4\omega})$$

$$= \mathcal{O}(\frac{\mu\ \sqrt{h}}{\theta^3 D^3}\bar{\varepsilon}_\omega + A_{4\omega} + \frac{h^2}{\theta^2 D^2} B_{4\omega})$$

so that

$$m_\omega = \mathcal{O}(\frac{\mu\ \bar{\varepsilon}_\omega}{\theta^2 D^2} + \frac{\theta D}{\sqrt{h}} A_{4\omega} + \frac{h\sqrt{h}}{\theta D} B_{4\omega}) .$$

Similarly

$$\frac{\sqrt{h}}{\theta D}\zeta_4|t''_\omega| = \mathcal{O}(B_{4\omega} + \frac{1}{\theta^2 D^2}\|t''_\omega\|_4) = \mathcal{O}(\frac{\mu\ \bar{\varepsilon}_\omega}{\sqrt{h}\ \theta^2 D^2} + B_{4\omega})$$

and

$$M_\omega = \mathcal{O}(\frac{\mu\ \bar{\varepsilon}_\omega}{h\ \theta\ D} + \frac{\theta D}{\sqrt{h}} B_{4\omega}) .$$

Combining this with (3.16), we have

$$m_1 = \mathcal{O}(\frac{\mu\ \varepsilon_1}{\theta^2 D^2} + \frac{\theta^2 D^2}{h^2}\varepsilon_1 m_2 + \theta^2 m_1 + \frac{\theta h}{D} M_1 + \frac{\theta D}{h}\varepsilon_1 M_2)$$

$$m_2 = \mathcal{O}[\frac{\mu\ \bar{\varepsilon}_2}{\theta^2 D^2} + \frac{\theta^2 D^2}{h^2}(\varepsilon_1+\bar{\varepsilon}_2) m_1 + \theta^2 m_2 + \frac{\theta D}{h}(\varepsilon_1+\bar{\varepsilon}_2) M_1 + \frac{\theta h}{D} M_2]$$

$$M_1 = \mathcal{O}\left[\frac{\mu\varepsilon_1}{h\theta D} + \theta^2 M_1 + \frac{\theta^2 D^2}{h^2}\varepsilon_1 M_2 + \frac{\theta^3 D}{h} m_1 + \frac{\theta^3 D^3}{h^3}\varepsilon_1 m_2 \right]$$

$$M_2 = \mathcal{O}\left[\frac{\mu\bar{\varepsilon}_2}{h\theta D} + \frac{\theta^2 D^2}{h^2}(\varepsilon_1+\bar{\varepsilon}_2)M_1 + \theta^2 M_2 + \frac{\theta^3 D^3}{h^3}(\varepsilon_1+\bar{\varepsilon}_2)m_1 + \frac{\theta^3 D}{h} m_2 \right] .$$

For θ sufficiently small we can eliminate m_ω, M_ω from the right-hand sides to obtain

$$m_\omega = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta^2 D^2} \right) , \qquad M_\omega = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h\theta D} \right) .$$

Substituting this in (3.16), we get (3.12) for k = 4.

The above also gives

$$\zeta_4 |\ddot{t}_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta^2 D^2} \right) , \qquad \zeta_4 |t''_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h\theta D} \right)$$

or

$$|\ddot{t}_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{\theta^2 D^2} \right) , \quad |t''_\omega| = \mathcal{O}\left(\frac{\mu\bar{\varepsilon}_\omega}{h\theta D} \right) \text{ where } |x_3| \leq h,\ \zeta_5 \neq 0.$$

As before using this, (2.1), and (3.10), we obtain (3.15) for k = 2.

When we differentiate (3.13) twice, we see that the result is linear in those derivatives for which we do not yet have pointwise estimates, namely those of order 3,4,5. This makes thenext step easier and we obtain

$$\begin{aligned}\zeta_5 |[(t_1 t_2 + t_1^3)F(t_1,t_2)]^{\cdot\cdot\cdot\cdot\prime}| = \mathcal{O}\Big[& \frac{h^2}{D^2}\zeta_5 |t_1^{\prime\prime\prime\prime\prime}| + \varepsilon_1 \zeta_5 |t_2^{\prime\prime\prime\prime\prime}| + \frac{h^2}{\theta D^3}\zeta_4 |t_1^{\prime\prime\prime\prime}| \\ & + \frac{\varepsilon_1}{\theta D}\zeta_4 |t_2^{\prime\prime\prime\prime}| + \frac{h}{D^2}\zeta_4 |\ddot{\ddot{t}}_1| + \frac{\varepsilon_1}{h}\zeta_4 |\ddot{\ddot{t}}_2| + \frac{h^2}{\theta^2 D^4}\zeta_3 |t_1^{\prime\prime\prime}| + \frac{\varepsilon_1}{\theta^2 D^2}\zeta_3 |t_2^{\prime\prime\prime}| \\ & + \frac{h}{\theta D^3}\zeta_3 |\dddot{t}_1| + \frac{\varepsilon_1}{h\theta D}\zeta_3 |\dddot{t}_2| + \frac{\mu h \varepsilon_1}{\theta^2 D^6} \Big]\end{aligned}$$

$$\zeta_5|[(t_1\ t_2+t_1^3)F(t_1,t_2)]'''''| = \mathcal{O}[\frac{h^2}{D^2}\zeta_5|t_1'''''|+\varepsilon_1\zeta_5|t_2'''''|+\frac{h}{D^2}\zeta_4|t_1''''|$$

$$+\frac{\varepsilon_1}{h}\zeta_4|t_2''''|+\frac{h}{\theta D^3}\zeta_3|t_1'''|+\frac{\varepsilon_1}{h\theta D}\zeta_3|t_2'''|+\frac{\mu h\varepsilon_1}{\theta^2 D^6}]$$

$$\zeta_5|[(t_1^2+t_2^2)F(t_1,t_2)^{\cdot\cdot\cdot\cdot\prime}| = \mathcal{O}\Big\{(\varepsilon_1+\bar{\varepsilon}_2)[\zeta_5|t_1''''|+\frac{1}{\theta D}\zeta_4|t_1''''|$$

$$+\frac{1}{h}\zeta_4|\ddddot{t}_1|+\frac{1}{\theta^2D^2}\zeta_3|t_1'''|+\frac{1}{h\theta D}\zeta_3|\dddot{t}_1|]+\frac{h^2}{D^2}\zeta_5|t_2'''''|$$

$$+\frac{h^2}{\theta D^3}\zeta_4|t_2''''|+\frac{h}{D^2}\zeta_4|\ddddot{t}_2|+\frac{h^2}{\theta^2D^4}\zeta_3|t_2'''|+\frac{h}{\theta D^3}\zeta_3|\dddot{t}_2|+\frac{\mu h\bar{\varepsilon}_2}{\theta^2D^6}\Big\}$$

$$\zeta_5|[(t_1^2+t_2^2)F(t_1,t_2)]'''''| = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)(\zeta_5|t_1'''''|+\frac{1}{h}\zeta_4|t_1''''|$$

$$+\frac{1}{h\theta D}\zeta_3|t_1'''|)+\frac{h^2}{D^2}\zeta_5|t_2'''''|+\frac{h}{D^2}\zeta_4|t_2''''|+\frac{h}{\theta D^3}\zeta_3|t_2'''|+\frac{\mu h\bar{\varepsilon}_2}{\theta^2D^6}]$$

valid where $\zeta_5 \neq 0$.

Thus using (3.12) for k = 3,4, we obtain

$$\|\zeta_5[(t_1t_2+t_1^3)F(t_1,t_2)]^{\cdot\cdot\cdot\cdot\prime}\|_5 = \mathcal{O}(\frac{h^2}{D^2}B_{51}+\varepsilon_1B_{52}+\frac{\mu h\sqrt{h}}{\theta^3D^5}\varepsilon_1)$$

$$\|\zeta_5[(t_1t_2+t_1^3)F(t_1,t_2)]'''''\|_5 = \mathcal{O}(\frac{h^2}{D^5}B_{51}+\varepsilon_1B_{52}+\frac{\mu\sqrt{h}}{\theta^2D^4}\varepsilon_1)$$

$$\|\zeta_5[(t_1^2+t_2^2)F(t_1,t_2)]^{\cdot\cdot\cdot\cdot\prime}\|_5 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)B_{51}+\frac{h^2}{D^2}B_{52}+\frac{\mu h\sqrt{h}}{\theta^3D^5}\bar{\varepsilon}_2]$$

$$\|\zeta_5[(t_1^2+t_2^2)F(t_1,t_2)]'''''\|_5 = \mathcal{O}[(\varepsilon_1+\bar{\varepsilon}_2)B_{51}+\frac{h^2}{D^2}B_{52}+\frac{\mu\sqrt{h}}{\theta^2D^4}\bar{\varepsilon}_2]\ .$$

Also from (3.13) and (3.12)

$$\| \overset{*\vee\cdot}{t}_{\omega} \|_5 = \mathcal{O}(A_{3\omega}) = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta^2 D^2} \bar{\varepsilon}_{\omega} \right)$$

$$\| [(t_1 t_2 + t_1^3) F(t_1, t_2)]''' \|_5 = \mathcal{O}\left(\frac{h^2}{D^2} B_{31} + \varepsilon_1 B_{32} + \frac{\mu\sqrt{h}\,\varepsilon_1}{D^2} \right) = \mathcal{O}\left(\frac{\mu\sqrt{h}\varepsilon_1}{D^2} \right)$$

$$\| [(t_1^2 + t_2^2) F(t_1, t_2)]''' \|_5 = \mathcal{O}\left[(\varepsilon_1 + \bar{\varepsilon}_2) B_{31} + \frac{h^2}{D^2} B_{32} + \frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{D^2} \right] = \mathcal{O}\left(\frac{\mu\sqrt{h}\,\bar{\varepsilon}_2}{D^2} \right).$$

Using the above in (3.6) and (3.7) gives

$$A_{51} = \mathcal{O}\left(\frac{\mu\sqrt{h}}{\theta^4 D^4} \varepsilon_1 + \frac{\theta h}{D} B_{51} + \frac{\theta D}{h} \varepsilon_1 B_{52} \right] + o\left(\frac{h}{\theta D} B_{51} \right)$$

$$A_{52} = \mathcal{O}\left[\frac{\mu\sqrt{h}}{\theta^4 D^4} \bar{\varepsilon}_2 + \frac{\theta D}{h} (\varepsilon_1 + \bar{\varepsilon}_2) B_{51} + \frac{\theta h}{D} B_{52} \right] + o\left(\frac{h}{\theta D} B_{52} \right)$$

$$B_{51} = \mathcal{O}\left(\frac{\mu\,\varepsilon_1}{\sqrt{h}\,\theta^3 D^3} + \frac{\theta D}{h} A_{51} + \theta^2 B_{51} + \varepsilon_1 B_{52} \right)$$

$$B_{52} = \mathcal{O}\left[\frac{\mu\,\bar{\varepsilon}_2}{\sqrt{h}\,\theta^3 D^3} + \frac{\theta D}{h} A_{52} + (\varepsilon_1 + \bar{\varepsilon}_2) B_{51} + \theta^2 B_{52} \right].$$

For θ sufficiently small, we obtain (3.12) for $k = 5$.

From the last line of (3.3), we again obtain (3.15) for $k = 3$.

We can continue this procedure indefinitely, subject only to the existence of sufficiently many derivatives. Hence, this completes the proof of (3.12) and (3.15).

The first result of (0.5) follows from (3.15) for $\omega = 1$.

If f is in b_{k2} but not in a_{k2}, then f is an odd function of x_3 so that $f = 0$ for $x_3 = 0$. Also $f_{,3}$ is in $a_{k+1,2}$. Thus

$$f = f|_{x_3=0} + x_3 f_{,3}|_{x_3=\bar{x}_3} \quad \text{(for some } \bar{x}_3 \text{ with } |\bar{x}_3| < h)$$

$$= \mathcal{O}(h|a_{k+1,2}|) = \mathcal{O}\left(\frac{\mu h\,\bar{\varepsilon}_2}{(\theta D)^{k+1}} \right).$$

Therefore, at points for which the undeformed distance to the edge is at least θD, $|b_{k2}| = \mathcal{O}[\frac{\mu h \bar{\varepsilon}_2}{(\theta D)^{k+1}}]$, if θ is sufficiently small and sufficiently many derivatives exist. The second result in (0.5) follows from this and from $|a_{k2}| = \mathcal{O}[\frac{\mu \varepsilon_2}{(\theta D)^k}]$.

Since $\frac{\partial^k t_{\alpha 31}}{\partial x_1^{k_1} \partial x_2^{k_2}} = 0$ for $x_3 = \pm h$, then

$$\frac{\partial^k t_{\alpha 31}}{\partial x_1^{k_1} \partial x_2^{k_2}} = (x_3 - h) \left. \frac{\partial^{k+1} t_{\alpha 31}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3} \right|_{x_3 = \bar{x}_3} \qquad \text{(for some } \bar{x}_3 \text{ with } |\bar{x}_3| < h)$$

$$= \mathcal{O}(h|a_{k+1,1}|) = \mathcal{O}[\frac{\mu h \varepsilon_1}{(\theta D)^{k+1}}]$$

which gives the third result in (0.5).

Since $\partial^k t_{\alpha 32}/\partial x_1^{k_1} \partial x_2^{k_2} = 0$ for $x_3 = -h, 0, h$, then

$\frac{\partial^{k+1} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3} = 0$ for two distinct values of x_3 and

$\frac{\partial^{k+2} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^2} = 0$ for some value of x_3 with $|x_3| < h$.

Thus for some $\bar{x}_3$,

$$\frac{\partial^{k+2} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^2} = (x_3 - \bar{x}_3) \left. \frac{\partial^{k+3} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^3} \right|_{x_3 = \hat{x}_3} = \mathcal{O}(h|a_{k+3,2}|)$$

$$= \mathcal{O}[\frac{\mu h \bar{\varepsilon}_2}{(\theta D)^{k+3}}].$$

Hence,

$$\frac{\partial^{k+1} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3} = \mathcal{O}\left[\frac{\mu\, h^2\, \bar{\varepsilon}_2}{(\theta D)^{k+3}} \right] ,$$

and

$$\frac{\partial^{k} t_{\alpha 32}}{\partial x_1^{k_1} \partial x_2^{k_2}} = \mathcal{O}\left[\frac{\mu\, h^3\, \bar{\varepsilon}_2}{(\theta D)^{k+3}} \right] .$$

This gives the fourth result of (0.5).

Since $\partial^k t_{33\omega}/\partial x_1^{k_1} \partial x_2^{k_2} = 0$ for $x_3 = \pm h$, then $\partial^{k+1} t_{33\omega}/\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3 = 0$ for some x_3 with $|x_3| < h$. Thus for some $\bar{x}_3$

$$\frac{\partial^{k+1} t_{33\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3} = (x_3 - \bar{x}_3) \left. \frac{\partial^{k+2} t_{33\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^2} \right|_{x_3 = \hat{x}_3} = \mathcal{O}(h|a_{k+2,\omega}|)$$

$$= \mathcal{O}\left[\frac{\mu h\, \bar{\varepsilon}_\omega}{(\theta D)^{k+2}} \right] ,$$

so that

$$\frac{\partial^{k} t_{33\omega}}{\partial x_1^{k_1} \partial x_2^{k_2}} = \mathcal{O}\left[\frac{\mu\, h^2 \bar{\varepsilon}_\omega}{(\theta D)^{k+2}} \right] .$$

This gives the last result in (0.5) and the proof of (0.5) is complete.

From (1.5) and (0.5) we easily obtain

$$\frac{\partial^k e_{ij\omega}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left[\dfrac{\varepsilon_\omega}{(\theta D)^k} \right] & \begin{cases} \text{if } k_3 = 0 \\ \text{if } k_3 \text{ is even \& } (i,j) = (\alpha,\beta) \text{ or } (3,3) \\ \text{if } k_3 \text{ is odd and } (i,j) = (\alpha,3) \end{cases} \\ \mathcal{O}\left[\dfrac{\varepsilon_\omega}{h(\theta\ D)^{k-1}} \right] & \text{otherwise} \end{cases}$$

For $\omega = 1$, this is the first result of (0.6).

The second result of (0.6) is obtained from the above by the same argument used for the second result of (0.5).

From (1.3)

$$t_{\alpha 3} = 2(\mu + Be_{kk} + \mathcal{A}_2)e_{\alpha 3} + (\mathcal{C} + \mathcal{A}_1)e_{\alpha k}e_{k3}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are analytic in s_1, s_2, s_3 and of order 1 and 2 respectively in the strains. For sufficiently small strains and stresses, these equations determine e_{13}, e_{23} as unique functions of t_{13}, t_{23}, e_{11}, e_{22}, e_{33}, e_{12}. For $t_{13} = t_{23} = 0$, this unique solution is clearly $e_{13} = e_{23} = 0$. Hence $e_{\alpha 3} = 0$ and also $e_{\alpha 3 \omega} = 0$ for $x_3 = \pm h$. The remaining results of (0.6) now follow using the same arguments used to obtain the third and fourth results of (0.5).

From (0.6) and the first line of (1.8) we obtain

$$(3.17)\quad \frac{\partial^{\ell}\Gamma_{ijk\omega}}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\partial x_3^{\ell_3}} = \begin{cases} \mathcal{O}\left[\dfrac{\bar{\varepsilon}_\omega}{(\theta D)^{\ell+1}}\right] & \text{if } \ell_3 \text{ plus the number of 3's among } i,j,k \text{ is even} \\ \mathcal{O}\left[\dfrac{\bar{\varepsilon}_\omega}{h(\theta D)^{\ell}}\right] & \text{otherwise} \end{cases}$$

which is valid for $\ell = 0,1,2,\ldots$ at points whose distance from the edge is at least θD.

From (1.7)

$$(3.18)\quad \begin{cases} g^{ij} = \delta_{ij} - 2e_{ij} + 4e_{ij}e_{kj} - 8e_{ik}e_{k\ell}e_{\ell j} + \cdots \\ g^{ij}{}_1 = -2e_{ij1} + 4(e_{ik1}e_{kj2} + e_{ik2}e_{kj1}) + \cdots = e_1 F(e_1, e_2) \\ g^{ij}{}_2 = \delta_{ij} - 2e_{ij2} + 4(e_{ik1}e_{kj1} + e_{ik2}e_{kj2}) + \cdots = \\ \qquad = \delta_{ij} + (e_2 + e_1^2)F(e_1, e_2) \end{cases}$$

Next we define Γ^k_{ij} in (3.19), and as usual we let $\Gamma^k_{ij\omega}$ be the even part of Γ^k_{ij} if ω plus the number of 3's among i,j,k is even, and we let $\Gamma^k_{ij\omega}$ be the odd part of Γ^k_{ij} otherwise ($\omega = 1,2$). We then have

$$(3.19)\begin{cases} \Gamma^k_{ij} = g^{k\ell}\,\Gamma_{ij\ell} \\[2ex] \Gamma^k_{ij1} = g^{k\ell}{}_1\Gamma_{ij\ell 2} + g^{k\ell}{}_2\Gamma_{ij\ell 1} = \Gamma_{ijk1} + (e_1e_2' + e_1'e_2 + e_1e_1')F(e_1,e_2) \\[2ex] \Gamma^k_{ij2} = g^{k\ell}{}_1\Gamma_{ij\ell 1} + g^{k\ell}{}_2\Gamma_{ij\ell 2} = \Gamma_{ijk2} + (e_2e_2' + e_1e_2' + e_1e_1')F(e_1,e_2) \\[2ex] \dfrac{\partial^\ell \Gamma^k_{ij\omega}}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\partial x_3^{\ell_3}} = \begin{cases} \mathcal{O}\left[\dfrac{\bar{\varepsilon}_\omega}{(\theta D)^{\ell+1}}\right] & \text{if } \ell_3 \text{ plus the number of 3's} \\ & \text{among } i,j,k \text{ is even} \\[2ex] \mathcal{O}\left[\dfrac{\bar{\varepsilon}_\omega}{h(\theta D)^{\ell}}\right] & \text{otherwise} \end{cases} \\[4ex] \dfrac{\partial^\ell \Gamma^k_{ij2}}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\partial x_3^{\ell_3}} = \mathcal{O}\left[\dfrac{h\bar{\varepsilon}_2}{(\theta D)^{\ell+2}}\right] \quad \text{if } \ell_3 \text{ plus the number of 3's} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{among } i,j,k \text{ is odd} \end{cases}$$

which is valid for $\ell = 0,1,2,\ldots$ at points whose distance from the edge is at least θD.

The second and third lines of (3.19) follow from the definition of $\Gamma^k_{ij\omega}$ and (3.18). The fourth line follows from the second and third lines using (3.17) and (0.6). The fifth line follows from the fourth line since in this case $\dfrac{\partial^\ell \Gamma^k_{ij2}}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \partial x_3^{\ell_3}}$ is an odd function of x_3 and

$$\frac{\partial^\ell \Gamma^k_{ij2}}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \partial x_3^{\ell_3}} = x_3 \left. \frac{\partial^{\ell+1} \Gamma^k_{ij2}}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \partial x_3^{\ell_3+1}} \right|_{x_3=\bar{x}_3} .$$

From (0.1) we have

$$e_{ij,k} = \frac{1}{2} (u_{i,jk} + u_{j,ik} + u_{\ell,ik} u_{\ell,j} + u_{\ell,i} u_{\ell,jk}) .$$

Permuting the indices and taking the appropriate linear combination, we get

$$(\delta_{i\ell}+u_{\ell,i})u_{\ell,jk} = \Gamma_{kji} .$$

But

$$\Gamma_{kji} = (\delta_{ij}+2e_{im})\Gamma^m_{kj} = (\delta_{\ell i}+u_{\ell,i})(\delta_{\ell m}+u_{\ell,m})\Gamma^m_{kj} .$$

Since the matrix $(\delta_{\ell i} + u_{\ell,i})$ is taken to be nonsingular, we get (3.20).

$$u_{\ell,jk} = (\delta_{\ell m} + u_{\ell,m})\Gamma^m_{ij} \; . \qquad (3.20)$$

We can pick our axes so that the origin is the undeformed position of a particle whose undeformed distance to the edge is at least θD. After a translation and rigid rotation of the deformed plate, we can have $u_i = 0$, $u_{\alpha,3} = 0$, and $u_{2,1} = 0$ at the origin.

As usual we consider deformations for which $\det(\delta_{ij}+u_{i,j}) > 0$ and which can be obtained by deforming continuously from the undeformed configuration in such a way that the determinant is always positive. Consequently, at the origin the positive determinant condition is

$$(1+u_{1,1})(1+u_{2,2})(1+u_{3,3}) > 0$$

and $1+u_{1,1} > 0$, $1+u_{2,2} > 0$, and $1+u_{3,3} > 0$ follows from the ability to maintin the positive determinant while deforming continuously from the undeformed state.

At the origin, $e_{33} = u_{3,3} + \frac{1}{2} u^2_{3,3}$ so that $u_{3,3} = -1 \pm \sqrt{1+2e_{33}}$. Since $1+u_{3,3} > 0$ at the origin, the plus sign is the right one and $u_{3,3} = -1+\sqrt{1+2e_{33}} = e_{33} + \mathcal{O}(e^2_{33}) = \mathcal{O}(\bar{\varepsilon}_2)$ at the origin.

At the origin $e_{\alpha 3} = \frac{1}{2}(u_{3,\alpha} + u_{3,\alpha}u_{3,3})$ so $u_{3,\alpha} = \mathcal{O}(\frac{h\varepsilon_1}{\theta D})$.

At the origin

$$e_{11} = u_{1,1} + \frac{1}{2}(u^2_{1,1} + u^2_{3,1}) = u_{1,1} + \frac{1}{2}u^2_{1,1} + \mathcal{O}\left[\frac{h^2}{(\theta D)^2}\varepsilon^2_1\right]$$

$$= u_{1,1} + \frac{1}{2}u^2_{1,1} + \mathcal{O}(\bar{\varepsilon}_2)$$

and $u_{1,1} = \mathcal{O}(\bar{\varepsilon}_2)$ since $1+u_{1,1} > 0$. Similarly $u_{2,2} = \mathcal{O}(\bar{\varepsilon}_2)$.

At the origin

$$e_{12} = \frac{1}{2}(u_{1,2}+u_{1,1}\, u_{1,2}) \qquad \text{so} \qquad u_{1,2} = \mathcal{O}(\bar{\varepsilon}_2) \ .$$

To summarize, at the origin

$$u_{i,j} = \begin{cases} \mathcal{O}(\bar{\varepsilon}_2) & \text{for } (i,j) = (\alpha,\beta) \text{ or } (3,3) \\ \mathcal{O}(\frac{h\varepsilon_1}{\theta D}) & \text{for } (i,j) = (\alpha,3). \end{cases}$$

Now consider a curve of particles on the middle surface which contains the particle which was at the origin in the undeformed plate. Let $x_\alpha = x_\alpha(s)$, $x_3 = 0$ be the equations of this curve of particles in the undeformed configuration where s is the arc length and require that the curve is at least θD units away from the edge. We assume the topology of the plate is simple enough so that all points of the undeformed middle surface, which are at least θD units from the edge, can be joined to the origin by such a curve whose length is less than KD where K is some universal constant. Then $s = \mathcal{O}(D)$ along such a curve. Let $w_{\ell k} = u_{\ell,k}$ on the middle surface and let $w_{\ell k0}$ be the value of $w_{\ell k}$ at the origin. Then from (3.20)

$$w_{\ell k,\alpha} = \Gamma^{\ell}_{k\alpha} + \Gamma^{m}_{k\alpha} w_{\ell m}$$

so that along the curve

$$\frac{dw_{\ell k}}{ds} = (\Gamma^{\ell}_{k\gamma} + \Gamma^{m}_{k\gamma} w_{\ell m})\, \frac{dx_\gamma}{ds}$$

and

$$w_{\ell k} = w_{\ell k0} + \int_0^s (\Gamma^{\ell}_{k\gamma} + \Gamma^{m}_{k\gamma} w_{\ell m})\, \frac{dx_\gamma}{ds}\, ds$$

where s = 0 at the origin.

Hence, using the above, the estimates at the origin, and (3.19), there is a constant K depending only on the material such that

$$|w_{\alpha\beta}| \leq K\{\bar{\varepsilon}_2 + \int_0^s [\,\frac{\bar{\varepsilon}_2}{\theta D}\,(1+|w_{\alpha 1}|+|w_{\alpha 2}|) + \frac{\varepsilon_1}{h}\,|w_{\alpha 3}|]\,ds\}$$

$$|w_{\alpha 3}| \leq K \int_0^s [\,\frac{\varepsilon_1}{h}\,(1+|w_{\alpha 1}|+|w_{\alpha 2}|) + \frac{\bar{\varepsilon}_2}{\theta D}\,|w_{\alpha 3}|]\,ds$$

$$|w_{3\alpha}| \leq K\left\{\frac{h\varepsilon_1}{\theta D} + \int_0^s [\frac{\varepsilon_1}{h}\,(1+|w_{33}|) + \frac{\bar{\varepsilon}_2}{\theta D}\,(|w_{31}|+|w_{32}|)]\,ds\right\}$$

$$|w_{33}| \leq K\left\{\bar{\varepsilon}_2 + \int_0^s [\frac{\bar{\varepsilon}_2}{\theta D}\,(1+|w_{33}|) + \frac{\varepsilon_1}{h}\,(|w_{31}|+|w_{32}|)]\,ds\right\} .$$

Let

$$w = \frac{\theta D}{h}\,\frac{\varepsilon_1}{\bar{\varepsilon}_2}\,(|w_{11}|+|w_{12}|+|w_{21}|+|w_{22}|+|w_{33}|)$$
$$+|w_{13}| + |w_{23}| + |w_{31}| + |w_{32}| .$$

From the above inequalities

$$w \leq K_1[\,\frac{D}{h}\,\varepsilon_1 + \frac{\theta}{D}\int_0^s w\,ds\,]$$

where K_1 is a constant depending only on the material.

Let $z = \int_0^s w\,ds$. Then

$$z' \leq K_1(\frac{D}{h}\,\varepsilon_1 + \frac{\theta}{D}\,z)$$

$$\left(z\,e^{-\frac{K_1\theta}{D}s}\right)_s \leq \frac{K_1 D}{h}\,\varepsilon_1\,e^{-\frac{K_1\theta}{D}s} .$$

Since z = 0 for s = 0, it follows that

$$z \leq \frac{D^2}{\theta h}\,\varepsilon_1(e^{\frac{K_1\theta}{D}s} - 1) = \mathcal{O}(\frac{D^2}{h}\,\varepsilon_1) .$$

Hence

$$w = z' = \mathcal{O}\left(\frac{D}{h}\,\varepsilon_1\right) ,$$

and from the definition of w

$$w_{\alpha\beta} = \mathcal{O}\left(\frac{1}{\theta}\,\bar{\varepsilon}_2\right) , \qquad w_{33} = \mathcal{O}\left(\frac{1}{\theta}\,\bar{\varepsilon}_2\right)$$

$$w_{\alpha 3} = \mathcal{O}\left(\frac{D}{h}\,\varepsilon_1\right) , \qquad w_{3\alpha} = \mathcal{O}\left(\frac{D}{h}\,\varepsilon_1\right)$$

That is

$$(3.21)\left\{\begin{array}{ll} u_{\alpha,\beta} = \mathcal{O}\left(\frac{\bar{\varepsilon}_2}{\theta}\right) , & u_{3,3} = \mathcal{O}\left(\frac{\bar{\varepsilon}_2}{\theta}\right) \\ & \\ u_{\alpha,3} = \mathcal{O}\left(\frac{D}{h}\,\varepsilon_1\right), & u_{3,\alpha} = \mathcal{O}\left(\frac{D}{h}\,\varepsilon_1\right) \end{array}\right\} \begin{array}{l} \text{if } x_3 = 0 \text{ and the undeformed} \\ \text{distance to the edge is} \\ \text{at least } \theta D. \end{array}$$

Now let $w_{\ell k} = u_{\ell,k}$ for $|x_3| \le h$. From (3.20)

$$w_{\ell k,3} = \Gamma^{\ell}_{k3} + \Gamma^{m}_{k3}\, w_{\ell m} .$$

As usual let $w_{\ell k\omega}$ be the even part of $w_{\ell k}$ if ω plus the number of 3's among ℓ,k is even, and let $w_{\ell k\omega}$ be the odd part otherwise. Then the above gives

$$w_{\ell k1,3} = \Gamma^{\ell}_{k31} + \Gamma^{m}_{k31}\, w_{\ell m2} + \Gamma^{m}_{k32}\, w_{\ell m1}$$

$$w_{\ell k2,3} = \Gamma^{\ell}_{k32} + \Gamma^{m}_{k31}\, w_{\ell m1} + \Gamma^{m}_{k32}\, w_{\ell m2} .$$

Integrating these from 0 to x_3 with respect to x_3 and using the initial conditions (3.21) and the estimates (3.19), we obtain

$$|w_{\alpha\beta 1}| \le K \int_0^{|x_3|} \Big[\frac{\varepsilon_1}{h}\,(1+|w_{\alpha 12}|+|w_{\alpha 22}|) + \frac{\varepsilon_1}{\theta D}\,|w_{\alpha 32}|$$

$$+ \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2(|w_{\alpha 11}|+|w_{\alpha 21}|) + \frac{\bar{\varepsilon}_2}{\theta D}\,|w_{\alpha 31}|\Big]\, dx_3$$

$$
\begin{aligned}
|w_{\alpha 31}| \le K\Big\{ \frac{D}{h}\,\varepsilon_1 + \int_0^{|x_3|} \Big[& \frac{\varepsilon_1}{\theta D}\,(1+|w_{\alpha 12}|+|w_{\alpha 22}|) + \frac{\varepsilon_1}{h}\,|w_{\alpha 32}| \\
& + \frac{\bar{\varepsilon}_2}{\theta D}\,(|w_{\alpha 11}|+|w_{\alpha 21}|) + \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2 |w_{\alpha 31}| \Big]\, dx_3 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
|w_{3\alpha 1}| \le K\Big\{ \frac{D}{h}\,\varepsilon_1 + \int_0^{|x_3|} \Big[& \frac{\varepsilon_1}{\theta D}\,(1+|w_{332}|) + \frac{\varepsilon_1}{h}\,(|w_{312}|+|w_{322}|) \\
& + \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2(|w_{311}|+|w_{321}|) + \frac{\bar{\varepsilon}_2}{\theta D}\,|w_{331}| \Big]\, dx_3 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
|w_{331}| \le K \int_0^{|x_3|} \Big[& \frac{\varepsilon_1}{h}\,(1+|w_{332}|) + \frac{\varepsilon_1}{\theta D}\,(|w_{312}|+|w_{322}|) \\
& + \frac{\bar{\varepsilon}_2}{\theta D}\,(|w_{311}|+|w_{321}|) + \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2 |w_{331}| \Big]\, dx_3
\end{aligned}
$$

$$
\begin{aligned}
|w_{\alpha\beta 2}| \le K\Big\{ \frac{\bar{\varepsilon}_2}{\theta} + \int_0^{|x_3|} \Big[& \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2(1+|w_{\alpha 12}|+|w_{\alpha 22}|) + \frac{\bar{\varepsilon}_2}{\theta D}\,|w_{\alpha 32}| \\
& + \frac{\varepsilon_1}{h}\,(|w_{\alpha 11}|+|w_{\alpha 21}|) + \frac{\varepsilon_1}{\theta D}\,|w_{\alpha 31}| \Big]\, dx_3 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
|w_{\alpha 32}| \le K \int_0^{|x_3|} \Big[& \frac{\bar{\varepsilon}_2}{\theta D}\,(1+|w_{\alpha 12}|+|w_{\alpha 22}|) + \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2 |w_{\alpha 32}| \\
& + \frac{\varepsilon_1}{\theta D}\,(|w_{\alpha 11}|+|w_{\alpha 21}|) + \frac{\varepsilon_1}{h}\,|w_{\alpha 31}| \Big]\, dx_3
\end{aligned}
$$

$$
\begin{aligned}
|w_{3\alpha 2}| \le K \int_0^{|x_3|} \Big[& \frac{\bar{\varepsilon}_2}{\theta D}\,(1+|w_{332}|) + \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2(|w_{312}| + |w_{322}|) \\
& + \frac{\varepsilon_1}{h}\,(|w_{311}| + |w_{321}|) + \frac{\varepsilon_1}{\theta D}\,|w_{331}| \Big]\, dx_3
\end{aligned}
$$

$$
\begin{aligned}
|w_{332}| \le K\Big\{ \frac{\bar{\varepsilon}_2}{\theta} + \int_0^{|x_3|} \Big[& \frac{h}{(\theta D)^2}\,\bar{\varepsilon}_2(1+|w_{332}|) + \frac{\bar{\varepsilon}_2}{\theta D}\,(|w_{312}|+|w_{322}|) \\
& + \frac{\varepsilon_1}{\theta D}\,(|w_{311}|+|w_{321}|) + \frac{\varepsilon_1}{h}\,|w_{331}| \Big]\, dx_3 \Big\} ,
\end{aligned}
$$

where K is a constant depending only on the material.

Let

$$w = \sum |w_{\alpha\beta 1}| + |w_{331}| + \frac{h}{\theta D} \sum (|w_{3\alpha 1}| + |w_{\alpha 31}|)$$

$$+ \frac{\varepsilon_1}{\bar{\varepsilon}_2} [\sum |w_{\alpha\beta 2}| + |w_{332}|] + \frac{h}{\theta D} \frac{\varepsilon_1}{\bar{\varepsilon}_2} \sum [|w_{\alpha 32}|+|w_{3\alpha 2}|].$$

The above inequalities give

$$w \leq K_1 [\frac{\varepsilon_1}{\theta} + \int_0^{|x_3|} (\frac{\varepsilon_1}{h} + \frac{\theta}{D} w) \, dx_3]$$

where K_1 is a constant depending only on the material.

Let $z = \int_0^{|x_3|} w \, dx_3$. Then

$$z \leq \frac{D\varepsilon_1}{\theta^2} [e^{\frac{K_1 \theta}{D}|x_3|} - 1] = \mathcal{O}(\frac{h\varepsilon_1}{\theta})$$

and

$$w = \mathcal{O}(\frac{\varepsilon_1}{\theta}) \ .$$

Considering the definition of w, we obtain

$$(3.22) \quad u_{\alpha\omega,\beta} = \mathcal{O}(\frac{\bar{\varepsilon}_\omega}{\theta}) \ , \quad u_{3\omega,3} = \mathcal{O}(\frac{\bar{\varepsilon}_\omega}{\theta}) \ , \quad u_{3\omega,\alpha} = \mathcal{O}(\frac{D\bar{\varepsilon}_\omega}{h}) \ , \quad u_{\alpha\omega,3} = \mathcal{O}(\frac{D\bar{\varepsilon}_\omega}{h})$$

The above are valid for $|x_3| \leq h$ if the undeformed distance to the edge is at least θD.

Now let u_{i0} be the displacements on the middle surface, and consider a curve on the middle surface which goes through the particle which was at the origin in the undeformed plate and stays θD units from the edge. Let $x_\gamma = x_\gamma(s)$ be the equations of the curve where s is arc length. Then $\frac{du_{i0}}{ds} = u_{i0,\gamma} \frac{dx_\gamma}{ds}$ along the curve so that

$$u_{i0} = \int_0^s u_{i0,\gamma} \frac{dx_\gamma}{ds} \, ds \ .$$

Hence,

$$|u_{\alpha 0}| \leq \int_0^s \frac{\bar{\varepsilon}_2}{\theta}\, ds = \mathcal{O}\left(\frac{D}{\theta}\, \bar{\varepsilon}_2\right)$$

and

$$|u_{30}| \leq \int_0^s \frac{D}{h}\, \varepsilon_1\, ds = \mathcal{O}\left(\frac{D^2}{h}\, \varepsilon_1\right) .$$

Since we are assuming that all particles on the middle surface, whose undeformed distance from the boundary is at least θD, can be joined to the origin by such a curve whose length is $\mathcal{O}(D)$, we have

$$u_{\alpha 0} = \mathcal{O}\left(\frac{D}{\theta}\, \bar{\varepsilon}_2\right) , \qquad u_{30} = \mathcal{O}\left(\frac{D^2}{h}\, \varepsilon_1\right) .$$

Next

$$u_{\alpha 1} = \int_0^{x_3} u_{\alpha 1,3}\, dx_3 , \qquad u_{31} = u_{30} + \int_0^{x_3} u_{31,3}\, dx_3$$

$$u_{\alpha 2} = u_{\alpha 0} + \int_0^{x_3} u_{\alpha 2,3}\, dx_3 , \qquad u_{32} = \int_0^{x_3} u_{32,3}\, dx_3 .$$

Using (3.22) and the estimates for u_{i0} , we have

$$(3.23) \quad \left\{ \begin{array}{ll} u_{\alpha 1} = \mathcal{O}(D\varepsilon_1) , & u_{31} = \mathcal{O}\left(\frac{D^2}{h}\, \varepsilon_1\right) \\ u_{\alpha 2} = \mathcal{O}\left(\frac{D}{\theta}\, \bar{\varepsilon}_2\right) , & u_{32} = \mathcal{O}\left(\frac{h}{\theta}\, \bar{\varepsilon}_2\right) . \end{array} \right.$$

From (3.20) we have

$$u_{\ell 1,jk} = \Gamma^{\ell}_{jk1} + \Gamma^{m}_{jk1}\, u_{\ell 2,m} + \Gamma^{m}_{jk2}\, u_{\ell 1,m}$$

$$u_{\ell 2,jk} = \Gamma^{\ell}_{jk2} + \Gamma^{m}_{jk1}\, u_{\ell 1,m} + \Gamma^{m}_{jk2}\, u_{\ell 2,m} .$$

From these and (3.19) we get (3.24) for $k = 2$. By an induction argument (3.24) follows for $k = 3,4,\ldots$.

$$(3.24)\qquad \frac{\partial^k u_{i\omega}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left[\dfrac{\bar{\varepsilon}_\omega}{(\theta D)^{k-1}}\right] & \text{if } i = 3 \text{ and } k_3 \text{ is odd, or} \\ & \text{if } i \neq 3 \text{ and } k_3 \text{ is even} \\ \mathcal{O}\left[\dfrac{\bar{\varepsilon}_\omega}{h(\theta D)^{k-2}}\right] & \text{otherwise} \end{cases}$$

As usual we use the fact that $\dfrac{\partial^k u_{i2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}$ is an odd function of x_3 in some cases and the estimate for $\dfrac{\partial^{k+1} u_{i2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3+1}}$ to obtain a sharper estimate. From (3.23), (3.22), and (3.24), we then have

$$(3.25)\begin{cases} u_{\alpha 1} = \mathcal{O}(D\varepsilon_1)\ , \quad u_{31} = \mathcal{O}(\frac{D^2}{h}\ \varepsilon_1) \\ u_{\alpha 2} = \mathcal{O}(\frac{D}{\theta}\ \bar{\varepsilon}_2)\ , \quad u_{32} = \mathcal{O}(\frac{h}{\theta}\ \bar{\varepsilon}_2) \\ u_{\alpha 1,\beta} = \mathcal{O}(\frac{\varepsilon_1}{\theta})\ , \quad u_{31,3} = \mathcal{O}(\frac{\varepsilon_1}{\theta}),\ u_{31,\alpha} = \mathcal{O}(\frac{D}{h}\ \varepsilon_1),\ u_{\alpha 1,3} = \mathcal{O}(\frac{D}{h}\ \varepsilon_1) \\ u_{\alpha 2,\beta} = \mathcal{O}(\frac{\bar{\varepsilon}_2}{\theta})\ , \quad u_{32,3} = \mathcal{O}(\frac{\bar{\varepsilon}_2}{\theta}),\ u_{32,\alpha} = \mathcal{O}(\frac{h}{\theta D}\bar{\varepsilon}_2),\ u_{\alpha 2,3} = \mathcal{O}(\frac{h}{\theta D}\bar{\varepsilon}_2) \\ \dfrac{\partial^k u_{i1}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left[\dfrac{\varepsilon_1}{(\theta D)^{k-1}}\right] & \text{if } i = 3\ (i \neq 3) \\ & \text{and } k_3 \text{ is odd (even)} \\ \mathcal{O}\left[\dfrac{\varepsilon_1}{h(\theta D)^{k-2}}\right] & \text{otherwise} \end{cases} \\ \dfrac{\partial^k u_{i2}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}} = \begin{cases} \mathcal{O}\left[\dfrac{\bar{\varepsilon}_2}{(\theta D)^{k-1}}\right] & \text{if } i = 3\ (i \neq 3) \\ & \text{and } k_3 \text{ is odd (even)} \\ \mathcal{O}\left[\dfrac{h\bar{\varepsilon}_2}{(\theta D)^{k}}\right] & \text{otherwise} \end{cases} \end{cases}$$

for $k = 2,3,\ldots$ valid where the undeformed distance to the edges is at least θD.

Since (0.7) follows immediately from this, the proof of (0.7) is completed.

4. Proof of (0.13) through (0.22').

From (0.7), $u_{31,3} = \mathcal{O}(\varepsilon_1)$. Integrating this from 0 to x_3, we obtain

$$u_{31} = w + \int_0^{x_3} u_{31,3}\,dx_3 = w + \mathcal{O}(h\varepsilon_1). \tag{4.1}$$

In all the results obtained in this section, it is important to notice that the result can be differentiated repeatedly with respect to x_1 or x_2 if the remainder estimate is divided by D for each differentiation, and the result can be differentiated with respect to x_3 if the remainder estimate is divided by h.

From the definitions of $u_{i\omega}$, $e_{ij\omega}$, and e_{ij} we have

$$\begin{aligned} e_{ij1} &= \frac{1}{2}\,(u_{i1,j} + u_{j1,i} + u_{k1,i}\,u_{k2,j} + u_{k2,i}\,u_{k1,j}) \\ e_{ij2} &= \frac{1}{2}\,(u_{i2,j} + u_{j2,i} + u_{k1,i}\,u_{k1,j} + u_{k2,i}\,u_{k2,j}) \,. \end{aligned} \tag{4.2}$$

From (4.2), (0.7), (0.6),

$$e_{\alpha31} = \frac{1}{2}\,(u_{\alpha1,3} + u_{31,\alpha}) + \mathcal{O}(\frac{h}{D}\,\varepsilon_1) = \mathcal{O}(\frac{h}{D}\,\varepsilon_1),$$

$$u_{\alpha1,3} = -\,w_{,\alpha} + \mathcal{O}(\frac{h}{D}\,\varepsilon_1)\ , \text{ and}$$

$$u_{\alpha1} = -\,x_3\,w_{,\alpha} + \mathcal{O}(\frac{h^2}{D}\,\varepsilon_1)\ . \tag{4.3}$$

Here and in the following, results derived previously will be used repeatedly without mentioning all of them each time.

From (0.7), $u_{\alpha2,3} = \mathcal{O}(\frac{h}{D}\,\bar{\varepsilon}_2)$ and

$$u_{\alpha2} = v_\alpha + \mathcal{O}(\frac{h^2}{D}\,\bar{\varepsilon}_2)\ . \tag{4.4}$$

From (4.2), etc.,

$$(4.5)\quad \begin{aligned} e_{\alpha\beta 1} &= -\,x_3\, w_{,\alpha\beta} + \mathcal{O}\left(\frac{h^2}{D^2}\,\varepsilon_1\right) \\ e_{\alpha\beta 2} &= \frac{1}{2}\,(v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha} w_{,\beta}) + \mathcal{O}\left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right). \end{aligned}$$

From (1.4) and (0.5),

$$\begin{aligned} t_{331} &= \lambda e_{\gamma\gamma 1} + (\lambda+2\mu) e_{331} + \mathcal{O}\left(\frac{\mu h^2}{D^2}\,\varepsilon_1\right) \\ &= \mathcal{O}\left(\frac{\mu h^2}{D^2}\,\varepsilon_1\right). \end{aligned}$$

This with (4.5) and (4.2) gives

$$(4.6)\quad \begin{aligned} e_{331} &= \frac{\lambda}{\lambda+2\mu}\, x_3\, \Delta w + \mathcal{O}\left(\frac{h^2}{D^2}\,\varepsilon_1\right) \\ u_{31,3} &= \frac{\lambda}{\lambda+2\mu}\, x_3\, \Delta w + \mathcal{O}\left(\frac{h^2}{D^2}\,\varepsilon_1\right) \\ u_{31} &= w + \frac{\lambda}{2(\lambda+2\mu)}\, x_3^2\, \Delta w + \mathcal{O}\left(\frac{h^3}{D^2}\,\varepsilon_1\right) . \end{aligned}$$

Similarly

$$(4.7)\quad \begin{aligned} e_{332} &= -\,\frac{\lambda}{\lambda+2\mu}\,(v_{\gamma,\gamma} + \frac{1}{2}\, w_{,\gamma} w_{,\gamma}) + \mathcal{O}\left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right) \\ u_{32,3} &= -\,\frac{\lambda}{\lambda+2\mu}\, v_{\gamma,\gamma} - \frac{\lambda+\mu}{\lambda+2\mu}\, w_{,\gamma} w_{,\gamma} + \mathcal{O}\left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right) \\ u_{32} &= -\,x_3\left(\frac{\lambda}{\lambda+2\mu}\, v_{\gamma,\gamma} + \frac{\lambda+\mu}{\lambda+2\mu}\, w_{,\gamma} w_{,\gamma}\right) + \mathcal{O}\left(\frac{h^3}{D^2}\,\bar{\varepsilon}_2\right) . \end{aligned}$$

From (4.5-7) and (1.4),

$$(4.8)\quad \begin{aligned} t_{\alpha\beta 1} &= -\,2\mu x_3\left(\frac{\lambda}{\lambda+2\mu}\,\Delta w\,\delta_{\alpha\beta} + w_{,\alpha\beta}\right) + \mathcal{O}\left(\frac{\mu h^2}{D^2}\,\varepsilon_1\right) \\ t_{\alpha\beta 2} &= 2\mu\left[\frac{\lambda}{\lambda+2\mu}\,(v_{\gamma,\gamma} + \frac{1}{2}\, w_{,\gamma} w_{,\gamma})\delta_{\alpha\beta} + \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha} w_{,\beta})\right] \\ &\qquad + \mathcal{O}\left(\frac{\mu h^2}{D^2}\,\bar{\varepsilon}_2\right). \end{aligned}$$

From (0.10) and (1.6), we have

(4.9) $$T_{\alpha\beta} = \frac{1}{4\mu h}\int_{-h}^{h}(t_{\alpha\beta} + p_{\alpha\beta})\,dx_3 = \frac{1}{4\mu h}\int_{-h}^{h}(t_{\alpha\beta 2} + p_{\alpha\beta 2})\,dx_3$$

so that

$$T_{\alpha\beta} = T_{\beta\alpha} \,,$$

and

$$T_{\alpha\beta,\beta} = \frac{1}{4\mu h}\int_{-h}^{h}(t_{\alpha\beta 2} + p_{\alpha\beta 2})_{,\beta}\,dx_3$$

$$= -\frac{1}{4\mu h}\int_{-h}^{h}(t_{\alpha 32} + p_{\alpha 32})_{,3}\,dx_3 = 0$$

Hence, a stress function ψ exists such that (0.11) holds.

From (1.6)

(4.10) $$p_{ij\omega} = \mathcal{O}\left(\frac{\mu h^2}{D^2}\,\bar{\varepsilon}_\omega\right) .$$

From (4.9), (4.10), and (4.8)

$$T_{\alpha\beta} = \frac{\lambda}{\lambda+2\mu}\left(v_{\gamma,\gamma} + \frac{1}{2}\,w_{,\gamma}w_{,\gamma}\right)\delta_{\alpha\beta} + \frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha}w_{,\beta}\right) + \left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right).$$

Summing this and using (0.11) we have

$$T_{\gamma\gamma} = \Delta\psi = \frac{3\lambda+2\mu}{\lambda+2\mu}\left(v_{\gamma,\gamma} + \frac{1}{2}\,w_{,\gamma}w_{,\gamma}\right) + \mathcal{O}\left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right),$$

so that

(4.11) $$\frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha}w_{,\beta}\right) = T_{\alpha\beta} - \frac{\lambda}{3\lambda+2\mu}\,\Delta\psi\delta_{\alpha\beta} + \mathcal{O}\left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right) .$$

Using (4.11) in (4.5), (4.7), and (4.8), we have

(4.12) $$\begin{cases} e_{\alpha\beta} = T_{\alpha\beta} - \dfrac{\lambda}{3\lambda+2\mu}\,\Delta\psi\,\delta_{\alpha\beta} + \mathcal{O}\left(\dfrac{h^2}{D^2}\,\bar{\varepsilon}_2\right) \\ e_{332} = -\dfrac{\lambda}{3\lambda+2\mu}\,\Delta\psi + \mathcal{O}\left(\dfrac{h^2}{D^2}\,\bar{\varepsilon}_2\right) \\ u_{32} = -x_3\left(\dfrac{\lambda}{3\lambda+2\mu}\,\Delta\psi + \dfrac{1}{2}\,w_{,\gamma}w_{,\gamma}\right) + \mathcal{O}\left(\dfrac{h^3}{D^2}\,\bar{\varepsilon}_2\right) \\ t_{\alpha\beta 2} = 2\mu T_{\alpha\beta} + \mathcal{O}\left(\dfrac{\mu h^2}{D^2}\,\bar{\varepsilon}_2\right). \end{cases}$$

From (4.11)

$$\frac{1}{2}(v_{\alpha,\beta\gamma}+v_{\beta,\alpha\gamma}+w_{,\alpha\gamma}w_{,\beta}+w_{,\alpha}w_{,\beta\gamma}) = T_{\alpha\beta,\gamma}-\frac{\lambda}{3\lambda+2\mu}\Delta\psi_{,\gamma}\delta_{\alpha\beta}+\mathcal{O}(\frac{h^2}{D^3}\bar{\varepsilon}_2).$$

Permuting α,β,γ and taking a linear combination, we obtain

$$(4.13)\quad v_{\alpha,\beta\gamma} = -w_{,\alpha}w_{,\beta\gamma}+T_{\alpha\beta,\gamma}+T_{\alpha\gamma,\beta}-T_{\beta\gamma,\alpha} - \frac{\lambda}{3\lambda+2\mu}(\delta_{\alpha\beta}\Delta\psi_{,\gamma}+\delta_{\alpha\gamma}\Delta\psi_{,\beta}-\delta_{\gamma\beta}\Delta\psi_{,\alpha})+\mathcal{O}(\frac{h^2}{D^3}\bar{\varepsilon}_2)$$

which determines v_α in terms of w and ψ up to a certain order and is line 1 of (0.13).

From (4.13),

$$v_{\alpha,\beta\gamma\nu} = -w_{,\alpha\nu}w_{,\beta\gamma}-w_{,\alpha}w_{,\beta\gamma\nu}+T_{\alpha\beta,\gamma\nu}+T_{\alpha\gamma,\beta\nu}-T_{\gamma\beta,\alpha\nu} - \frac{\lambda}{3\lambda+2\mu}(\delta_{\alpha\beta}\Delta\psi_{,\gamma\nu}+\delta_{\alpha\gamma}\Delta\psi_{,\beta\nu}-\delta_{\gamma\beta}\Delta\psi_{,\alpha\nu})+\mathcal{O}(\frac{h^2}{D^4}\bar{\varepsilon}_2)\ .$$

Interchanging γ and ν and subtracting, we have

$$0 = -w_{,\alpha\nu}w_{,\beta\gamma}+w_{,\alpha\gamma}w_{,\beta\nu}+T_{\alpha\gamma,\beta\nu}-T_{\alpha\nu,\beta\gamma}-T_{\gamma\beta,\alpha\nu}+T_{\nu\beta,\alpha\gamma} - \frac{\lambda}{3\lambda+2\mu}(\delta_{\alpha\gamma}\Delta\psi_{,\beta\nu}-\delta_{\alpha\nu}\Delta\psi_{,\beta\gamma}-\delta_{\gamma\beta}\Delta\psi_{,\alpha\nu}+\delta_{\nu\beta}\Delta\psi_{,\alpha\gamma})+\mathcal{O}(\frac{h^2}{D^4}\bar{\varepsilon}_2).$$

Letting $\alpha=\gamma$ and $\beta=\nu$ and summing over γ and ν, we obtain

$$(4.14)\quad \Delta^2\psi = \frac{3\lambda+2\mu}{4(\lambda+\mu)}[w_{,\gamma\nu}w_{,\gamma\nu}-(\Delta w)^2]+\mathcal{O}(\frac{h^2}{D^4}\bar{\varepsilon}_2)\ ,$$

where $\Delta^2 = \frac{\partial^4}{\partial x_1^4}+2\frac{\partial^4}{\partial x_1^2\partial x_2^2}+\frac{\partial^4}{\partial x_2^4}$. (4.14) is one of the plate equations to lower order than shown in (0.22).

From (1.6), (4.10), (4.8),

$$(t_{\alpha 31}+p_{\alpha 31})_{,3} = -t_{\alpha\beta 1,\beta}+\mathcal{O}(\frac{\mu h^2}{D^3}\varepsilon_1) = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}x_3\Delta w_{,\alpha}+\mathcal{O}(\frac{\mu h^2}{D^3}\varepsilon_1),$$

so that integrating from -h to x_3 gives

$$t_{\alpha 31} + p_{\alpha 31} = \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(x_3^2-h^2)\Delta w_{,\alpha} + \mathcal{O}\left(\frac{\mu h^3}{D^3}\varepsilon_1\right) . \tag{4.15}$$

From (1.6)

$$p_{\alpha 31} = \mathcal{O}\left(\frac{\mu h^3}{D^3}\varepsilon_1\right)$$

so that with (1.4), (4.2) for $e_{\alpha 31}$, we have

$$\left\{\begin{aligned} t_{\alpha 31} &= \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(x_3^2-h^2)\Delta w_{,\alpha} + \mathcal{O}\left(\frac{\mu h^3}{D^3}\varepsilon_1\right) \\ &= 2\mu e_{\alpha 31} + \mathcal{O}\left(\frac{\mu h^3}{D^3}\varepsilon_1\right) \\ e_{\alpha 31} &= \frac{\lambda+\mu}{\lambda+2\mu}(x_3^2-h^2)\Delta w_{,\alpha} + \mathcal{O}\left(\frac{h^3}{D^3}\varepsilon_1\right) \\ u_{\alpha 1} &= -x_3 w_{,\alpha} + \left[\frac{3\lambda+4\mu}{6(\lambda+2\mu)}x_3^3 - \frac{2(\lambda+\mu)}{\lambda+2\mu}x_3 h^2\right]\Delta w_{,\alpha} \\ &\quad + x_3 w_{,\alpha}\left(\frac{\lambda}{3\lambda+2\mu}\Delta\psi + \frac{1}{2}w_{,\gamma}w_{,\gamma}\right) + x_3 w_{,\gamma}v_{\gamma,\alpha} + \mathcal{O}\left(\frac{h^4}{D^3}\varepsilon_1\right) . \end{aligned}\right. \tag{4.16}$$

Similarly, since $e_{\alpha 32} = \mathcal{O}\left(\frac{h^3}{D^3}\bar{\varepsilon}_2\right)$, from (4.2) we obtain

$$u_{\alpha 2} = v_\alpha + x_3^2\left[\frac{\lambda}{2(3\lambda+2\mu)}\Delta\psi_{,\alpha} - \frac{\lambda}{2(\lambda+2\mu)}w_{,\alpha}\Delta w\right] + \mathcal{O}\left(\frac{h^4}{D^3}\bar{\varepsilon}_2\right) . \tag{4.17}$$

From (4.15)

$$\begin{aligned} (t_{331}+p_{331})_{,3} &= -(t_{3\alpha 1}+p_{3\alpha 1})_{,\alpha} \\ &= -\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(x_3^2-h^2)\Delta^2 w + \mathcal{O}\left(\frac{\mu h^3}{D^4}\varepsilon_1\right) . \end{aligned} \tag{4.18}$$

From (1.6), (1.1), (1.4)

$$
(4.19)\left\{\begin{aligned}
p_{331} &= -W_o + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\varepsilon_1\right) \\
s_{1e} &= \frac{2\mu}{3\lambda+2\mu}\,\Delta\psi + \mathcal{O}\left(\frac{h^2}{D^2}\,\bar{\varepsilon}_2\right) \\
s_{1o} &= -\frac{2\mu}{\lambda+2\mu}\,x_3\Delta w + \mathcal{O}\left(\frac{h^2}{D^2}\,\varepsilon_1\right) \\
s_{2o} &= \frac{4\lambda\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\,x_3\Delta\psi\;\Delta w - 2x_3T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{h^4}{D^4}\,\varepsilon_1\right) \\
W_o &= -2\mu x_3T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\varepsilon_1\right) \\
p_{331,3} &= 2\mu T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{\mu h^3}{D^4}\,\varepsilon_1\right) .
\end{aligned}\right.
$$

From (4.18) and (4.19)

$$
t_{331,3} = -2\mu T_{\gamma\nu}w_{,\gamma\nu} - \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(x_3^2-h^2)\Delta^2 w + \mathcal{O}\left(\frac{\mu h^3}{D^4}\,\varepsilon_1\right) .
$$

Integrating from $-h$ to x_3 ,

$$
t_{331} = -2\mu(x_3+h)T_{\gamma\nu}w_{,\gamma\nu} - \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{1}{3}\,x_3^3 - \frac{2}{3}\,h^3 - x_3h^2\right)\Delta^2 w + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\varepsilon_1\right) .
$$

Since $t_{331} = 0$ for $x_3 = h$, the above implies

$$
(4.20)\quad \frac{2(\lambda+\mu)}{3(\lambda+2\mu)}\,h^2\Delta^2 w = T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{h^3}{D^4}\,\varepsilon_1\right)
$$

which is another of the plate equations to lower order than shown in (0.22).

Using (4.20) in the above expression for t_{331} gives

$$
(4.21)\quad t_{331} = \mu x_3\left(1-\frac{x_3^2}{h^2}\right)T_{\gamma\nu}w_{,\gamma\nu} + \mathcal{O}\left(\frac{h^4}{D^4}\,\varepsilon_1\right) .
$$

Using the formulas for replacements derived so far in section 4 in (4.2), we obtain line 1 of (0.16).

Using line one of (0.16) in (1.4) we have

$$t_{331} = (\lambda+2\mu)e_{331} + x_3\Big\{-\lambda\Delta w + [\frac{\lambda(3\lambda+4\mu)}{4(\lambda+\mu)}\,\frac{x_3^2}{h^2} - 3\lambda - 2B]T_{\gamma\nu}w_{,\gamma\nu}$$

$$- \lambda(w_{,\gamma}\Delta\psi_{,\gamma} + \frac{1}{2}w_{,\gamma}w_{,\gamma}\,\Delta w)$$

$$+ [\frac{\lambda^2}{3\lambda+2\mu} - \frac{8\mu^2A - 12\lambda\mu B + 2\lambda^2C}{(3\lambda+2\mu)(\lambda+2\mu)}]\,\Delta\psi\,\Delta w\Big\} + \mathcal{O}(\frac{\mu h^4}{D^4}\,\varepsilon_1).$$

From this and (4.21) we obtain line 1 of (0.18) where

$$(4.22)\begin{cases} C_1 = \dfrac{3\lambda+\mu+2B}{\lambda+2\mu} \\[2ex] C_2 = -\dfrac{3\lambda+2\mu}{4(\lambda+\mu)} \\[2ex] C_3 = -\dfrac{\lambda^2}{(3\lambda+2\mu)(\lambda+2\mu)} + \dfrac{8\mu^2A - 12\lambda\mu B + 2\lambda^2C}{(3\lambda+2\mu)(\lambda+2\mu)^2} \end{cases}$$

From (4.2)

$$e_{331} = u_{31,3} + x_3\Big\{-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi_{,\gamma}w_{,\gamma} + \frac{\lambda}{2(\lambda+2\mu)}\,w_{,\gamma}w_{,\gamma}\Delta w$$

$$- \frac{\lambda^2}{(\lambda+2\mu)(3\lambda+2\mu)}\,\Delta\psi\,\Delta w\Big\} + \mathcal{O}(\frac{h^4}{D^4}\,\varepsilon_1)\ .$$

From this and line 1 of (0.18), we obtain $u_{31,3}$ which we integrate to obtain line 1 of (0.15) where

$$(4.23)\begin{cases} C_4 = \dfrac{4\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\[2ex] C_5 = C_3 + \dfrac{\lambda^2}{(\lambda+2\mu)(3\lambda+2\mu)} = \dfrac{8\mu^2A - 12\lambda\mu B + 2\lambda^2C}{(3\lambda+2\mu)(\lambda+2\mu)^2} \end{cases}$$

From (1.6)

$$p_{3\alpha 2} = \mathcal{O}(\frac{\mu h^3}{D^3}\bar{\varepsilon}_2)$$

so that

$$(4.24)\quad (t_{332} + p_{332})_{,3} = -(t_{3\alpha 2} + p_{3\alpha 2})_{,\alpha} = \mathcal{O}(\frac{\mu h^3}{D^4}\bar{\varepsilon}_2) \ .$$

From (1.6), (1.1), (4.19), (1.4)

$$(4.25)\left\{\begin{aligned}
p_{332} &= -\,W_e + \mathcal{O}(\frac{\mu h^4}{D^4}\bar{\varepsilon}_2)\\
s_{2e} &= \psi_{,\gamma\nu}\psi_{,\gamma\nu} - \frac{\lambda(3\lambda+4\mu)}{(3\lambda+2\mu)^2}(\Delta\psi)^2 + x_3^2[w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda^2}{(\lambda+2\mu)^2}(\Delta w)^2]\\
&\qquad + \mathcal{O}(\frac{h^4}{D^4}\bar{\varepsilon}_2)\\
W_e &= \mu\psi_{,\gamma\nu}\psi_{,\gamma\nu} - \frac{\lambda\mu}{3\lambda+2\mu}(\Delta\psi)^2 + x_3^2[\mu w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda\mu}{\lambda+2\mu}(\Delta w)^2]\\
&\qquad + \mathcal{O}(\frac{\mu h^4}{D^4}\bar{\varepsilon}_2)\\
p_{332,3} &= -2\mu x_3[w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda}{\lambda+2\mu}(\Delta w)^2] + \mathcal{O}(\frac{\mu h^3}{D^4}\bar{\varepsilon}_2) \ .
\end{aligned}\right.$$

From (4.24) and (4.25)

$$t_{332,3} = 2\mu x_3[w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda}{\lambda+2\mu}(\Delta w)^2] + \mathcal{O}(\frac{\mu h^3}{D^4}\bar{\varepsilon}_2) \ .$$

Integrating from $-h$ to x_3 gives

$$(4.26)\quad t_{332} = \mu(x_3^2-h^2)[w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda}{\lambda+2\mu}(\Delta w)^2] + \mathcal{O}(\frac{\mu h^4}{D^4}\bar{\varepsilon}_2) \ .$$

From the formulas derived thus far and from (4.2)

$$(4.27)\quad e_{\alpha\beta 2} = \frac{1}{2}\Big\{v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha}w_{,\beta} + v_{\gamma,\alpha}v_{\gamma,\beta}$$

$$+ x_3^2[\frac{\lambda}{3\lambda+2\mu}\Delta\psi_{,\alpha\beta} - \frac{\lambda}{\lambda+2\mu}w_{,\alpha\beta}\Delta w + w_{,\gamma\alpha}w_{,\gamma\beta}]\Big\} + \mathcal{O}(\frac{h^4}{D^4}\bar{\varepsilon}_2) \ .$$

From (1.4),

$$t_{332} = \lambda e_{\gamma\gamma 2} + (\lambda+2\mu)e_{332} + B\psi_{,\gamma\nu}\psi_{,\gamma\nu} + \frac{4\mu^2A-\lambda(3\lambda+8\mu)B+\lambda^2C}{(3\lambda+2\mu)^2}(\Delta\psi)^2$$

$$+ Bx_3^2 w_{,\gamma\nu}w_{,\gamma\nu} + \frac{4\mu^2A+\lambda(\lambda-4\mu)B+\lambda^2C}{(\lambda+2\mu)^2}x_3^2(\Delta w)^2 + \mathcal{O}\left(\frac{\mu h^4}{D^4}\bar{\varepsilon}_2\right).$$

From this and (4.26)

$$(4.28)\quad e_{332} = -\frac{\lambda}{\lambda+2\mu}\left[v_{\gamma,\gamma} + \frac{1}{2}w_{,\gamma}w_{,\gamma} + \frac{1}{2}v_{\gamma,\nu}v_{\gamma,\nu}\right] - \frac{B}{\lambda+2\mu}\psi_{,\gamma\nu}\psi_{,\gamma\nu}$$

$$+C_6(\Delta\psi)^2 + \left(C_7x_3^2 - \frac{\mu h^2}{\lambda+2\mu}\right)w_{,\gamma\nu}w_{,\gamma\nu} + \left[C_8x_3^2 - \frac{\lambda\mu}{(\lambda+2\mu)^2}h^2\right](\Delta w)^2 + \mathcal{O}\left(\frac{h^4}{D^4}\bar{\varepsilon}_2\right)$$

where

$$(4.29)\left\{\begin{aligned} C_6 &= -\frac{4\mu^2A-\lambda(3\lambda+8\mu)B+\lambda^2C}{(\lambda+2\mu)(3\lambda+2\mu)^2} \\ C_7 &= \frac{-5\lambda^2+4\lambda\mu+8\mu^2}{8(\lambda+\mu)(\lambda+2\mu)} - \frac{B}{\lambda+2\mu} \\ C_8 &= \frac{\lambda(5\lambda+4\mu)}{8(\lambda+\mu)(\lambda+2\mu)} - \frac{4\mu^2A+\lambda(\lambda-4\mu)B+\lambda^2C}{(\lambda+2\mu)^3}. \end{aligned}\right.$$

From (4.27), (4.28), (1.4), (1.6)

$$(4.30)\left\{\begin{aligned} e_{kk2} &= \frac{2\mu}{\lambda+2\mu}\left[v_{\gamma,\gamma} + \frac{1}{2}(w_{,\gamma}w_{,\gamma}+v_{\gamma,\nu}v_{\gamma,\nu})\right] - \frac{B}{\lambda+2\mu}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_6(\Delta\psi)^2 \\ &\quad + \left(C_9x_3^2 - \frac{\mu}{\lambda+2\mu}h^2\right)w_{,\gamma\nu}w_{,\gamma\nu} + \left[C_{10}x_3^2 - \frac{\lambda\mu}{(\lambda+2\mu)^2}h^2\right](\Delta w)^2 + \mathcal{O}\left(\frac{h^4}{D^4}\bar{\varepsilon}_2\right) \\ t_{\alpha\beta 2} &= \delta_{\alpha\beta}\Big\{\frac{2\lambda\mu}{\lambda+2\mu}\left[v_{\gamma,\gamma} + \frac{1}{2}(w_{,\gamma}w_{,\gamma}+v_{\gamma,\nu}v_{\gamma,\nu})\right] + \frac{2\mu B}{\lambda+2\mu}\psi_{,\gamma\nu}\psi_{,\gamma\nu} \\ &\quad + C_{11}(\Delta\psi)^2 + \left(C_{12}x_3^2 - \frac{\lambda\mu}{\lambda+2\mu}h^2\right)w_{,\gamma\nu}w_{,\gamma\nu} + \left[C_{13}x_3^2 - \frac{\lambda^2\mu}{(\lambda+2\mu)^2}h^2\right] \\ &\quad \cdot(\Delta w)^2\Big\} + \mu(v_{\alpha,\beta}+v_{\beta,\alpha}+w_{,\alpha}w_{,\beta} + v_{\gamma,\alpha}v_{\gamma,\beta}) + C_{14}T_{\alpha\beta}\Delta\psi + CT_{\alpha\gamma}T_{\gamma\beta} + \end{aligned}\right.$$

$$
\left|\begin{aligned}
&\quad + x_3^2\left[\frac{\lambda\mu}{3\lambda+2\mu}\,\Delta\psi_{,\alpha\beta} + C_{15}w_{,\alpha\beta}\Delta w + C_{16}w_{,\alpha\gamma}w_{,\gamma\beta}\right] + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\bar{\varepsilon}_2\right)\\
&p_{\alpha\beta 2} = \delta_{\alpha\beta}\left\{ -\mu\psi_{,\gamma\nu}\psi_{,\gamma\nu} + \frac{\lambda\mu}{3\lambda+2\mu}\,(\Delta\psi)^2 - \mu x_3^2\left[w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda}{\lambda+2\mu}(\Delta w)^2\right]\right\}\\
&\quad + 4\mu T_{\alpha\gamma}T_{\gamma\beta} - \frac{4\lambda\mu}{3\lambda+2\mu}\,T_{\alpha\beta}\Delta\psi + 4\mu x_3^2\left[w_{,\alpha\gamma}w_{,\gamma\beta} + \frac{\lambda}{\lambda+2\mu}\,w_{,\alpha\beta}\Delta w\right]\\
&\quad + \mathcal{O}\left(\frac{\mu h^4}{D^4}\,\bar{\varepsilon}_2\right)
\end{aligned}\right.
$$

where

$$
(4.31)\left\{\begin{aligned}
C_9 &= C_7 + \frac{5\lambda+4\mu}{8(\lambda+\mu)} = \frac{\mu(9\lambda+8\mu)}{4(\lambda+\mu)(\lambda+2\mu)} - \frac{B}{\lambda+2\mu}\\
C_{10} &= C_8 - \frac{\lambda(5\lambda+6\mu)}{8(\lambda+\mu)(\lambda+2\mu)} = \frac{-\lambda\mu}{4(\lambda+\mu)(\lambda+2\mu)} - \frac{4\mu^2A+\lambda(\lambda-4\mu)B+\lambda^2C}{(\lambda+2\mu)^3}\\
C_{11} &= \lambda C_6 + \frac{4\mu^2A-\lambda(3\lambda+8\mu)B+\lambda^2C}{(3\lambda+2\mu)^2} = 2\mu\,\frac{4\mu^2A-\lambda(3\lambda+8\mu)B+\lambda^2C}{(\lambda+2\mu)(3\lambda+2\mu)^2}\\
C_{12} &= \lambda C_9 + B = \frac{\lambda\mu(9\lambda+8\mu)}{4(\lambda+\mu)(\lambda+2\mu)} + \frac{2\mu B}{\lambda+2\mu}\\
C_{13} &= \lambda C_{10} + \frac{4\mu^2A+\lambda^2B}{(\lambda+2\mu)^2} = \frac{-\lambda^2\mu}{4(\lambda+\mu)(\lambda+2\mu)} + \frac{8\mu^3A+6\lambda^2\mu B-\lambda^3C}{(\lambda+2\mu)^3}\\
C_{14} &= \frac{4\mu B-2\lambda C}{3\lambda+2\mu}\\
C_{15} &= \frac{4\mu B-\lambda\mu}{\lambda+2\mu}\\
C_{16} &= \mu+C
\end{aligned}\right.
$$

From (4.9) and (4.30),

$$(4.32)\quad \begin{aligned} T_{\alpha\beta} = {} & \delta_{\alpha\beta}\Big\{\frac{\lambda}{\lambda+2\mu}\,[v_{\gamma,\gamma} + \tfrac{1}{2}(w_{,\gamma}w_{,\gamma} + v_{\gamma,\nu}v_{\gamma,\nu})] + C_{17}\psi_{,\gamma\nu}\psi_{,\gamma\nu} \\ & + C_{18}(\Delta\psi)^2 + h[C_{19}w_{,\gamma\nu}w_{,\gamma\nu} + C_{20}(\Delta w)^2]\Big\} \\ & + \tfrac{1}{2}\,(v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha}w_{,\beta} + v_{\gamma,\alpha}v_{\gamma,\beta}) + C_{21}T_{\alpha\beta}\Delta\psi + C_{22}T_{\alpha\gamma}T_{\gamma\beta} \\ & + h^2[\frac{\lambda}{6(3\lambda+2\mu)}\,\Delta\psi_{,\alpha\beta} + C_{23}w_{,\alpha\beta}\Delta w + C_{24}w_{,\alpha\gamma}w_{,\gamma\beta}] + \mathcal{O}(\frac{h^4}{D^4}\,\bar{\varepsilon}_2) \end{aligned}$$

where

$$(4.33)\left\{ \begin{aligned} C_{17} &= -\frac{1}{2} + \frac{B}{\lambda+2\mu} \\ C_{18} &= \frac{C_{11}}{2\mu} + \frac{\lambda}{2(3\lambda+2\mu)} &&= \frac{\lambda}{2(3\lambda+2\mu)} + \frac{4\mu^2 A - \lambda(3\lambda+8\mu)B + \lambda^2 C}{(\lambda+2\mu)(3\lambda+2\mu)^2} \\ C_{19} &= \frac{C_{12}}{6\mu} - \frac{2\lambda+\mu}{3(\lambda+2\mu)} &&= -\frac{7\lambda^2+16\lambda\mu+8\mu^2}{24(\lambda+\mu)(\lambda+2\mu)} + \frac{B}{3(\lambda+2\mu)} \\ C_{20} &= \frac{C_{13}}{6\mu} - \frac{\lambda(2\lambda+\mu)}{3(\lambda+2\mu)^2} &&= -\frac{\lambda(17\lambda^2+26\lambda\mu+8\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)^2} + \frac{8\mu^3 A + 6\lambda^2\mu B - \lambda^3 C}{6\mu(\lambda+2\mu)^3} \\ C_{21} &= \frac{C_{14}}{2\mu} - \frac{2\lambda}{3\lambda+2\mu} &&= \frac{2B-2\lambda}{3\lambda+2\mu} - \frac{\lambda C}{\mu(3\lambda+2\mu)} \\ C_{22} &= \frac{C}{2\mu} + 2 \\ C_{23} &= \frac{C_{15}}{6\mu} + \frac{2\lambda}{3(\lambda+2\mu)} &&= \frac{4B+3\lambda}{6(\lambda+2\mu)} \\ C_{24} &= \frac{C_{16}}{6\mu} + \frac{2}{3} &&= \frac{5}{6} + \frac{C}{6\mu} \end{aligned}\right.$$

Setting $\alpha = \beta$ in (4.32) and summing gives

$$(4.34)\quad \begin{aligned} v_{\gamma,\gamma} + \tfrac{1}{2}(w_{,\gamma}w_{,\gamma} + v_{\gamma,\nu}v_{\gamma,\nu}) = {} & \frac{\lambda+2\mu}{3\lambda+2\mu}\,\Delta\psi + C_{25}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{26}(\Delta\psi)^2 \\ & + h^2[C_{27}w_{,\gamma\nu}w_{,\gamma\nu} + C_{28}(\Delta w)^2] + \mathcal{O}(\frac{h^4}{D^4}\,\bar{\varepsilon}_2) \end{aligned}$$

where

$$
\left\{
\begin{aligned}
C_{25} &= -\frac{\lambda+2\mu}{3\lambda+2\mu}\,(2C_{17}+C_{22}) = -\frac{\lambda+2\mu}{3\lambda+2\mu} - \frac{2B}{3\lambda+2\mu} - \frac{\lambda+2\mu}{2\mu(3\lambda+3\mu)}\,C \\
C_{26} &= -\frac{\lambda+2\mu}{3\lambda+2\mu}\,(2C_{18}+C_{21}) = \frac{\lambda(\lambda+2\mu)}{(3\lambda+2\mu)^2} - \frac{8\mu^3(A+B)-\lambda(3\lambda^2+6\lambda\mu+4\mu^2)C}{\mu(3\lambda+2\mu)^3} \\
C_{27} &= -\frac{\lambda+2\mu}{3\lambda+2\mu}\,[2C_{19}+C_{24}+\frac{\lambda}{24(\lambda+\mu)}] \\
&= -\frac{7\lambda^2+30\lambda\mu+24\mu^2}{24(\lambda+\mu)(3\lambda+2\mu)} - \frac{2B}{3(3\lambda+2\mu)} - \frac{\lambda+2\mu}{6\mu(3\lambda+2\mu)}\,C \\
C_{28} &= -\frac{\lambda+2\mu}{3\lambda+2\mu}\,[2C_{20}+C_{23}-\frac{\lambda}{24(\lambda+\mu)}] \\
&= \frac{\lambda(23\lambda^2+20\lambda\mu-4\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} - \frac{8\mu^3A+8\mu(\lambda^2+\lambda\mu+\mu^2)B-\lambda^3C}{3\mu(\lambda+2\mu)^2(3\lambda+2\mu)}
\end{aligned}
\right.
\tag{4.35}
$$

Using (4.34) in (4.32), we obtain

$$\frac{1}{2}\,(v_{\alpha,\beta}+v_{\beta,\alpha}+w_{,\alpha}w_{,\beta}+v_{\gamma,\alpha}v_{\gamma,\beta}) = S_{\alpha\beta} + \mathcal{O}\,(\frac{h^4}{D^4}\,\bar{\varepsilon}_2)$$

where $S_{\alpha\beta}$ is defined in (0.13) and

$$
\left\{
\begin{aligned}
C_{29} &= -\frac{\lambda C_{25}}{\lambda+2\mu} - C_{17} = \frac{5\lambda+2\mu}{2(3\lambda+2\mu)} - \frac{B}{3\lambda+2\mu} + \frac{\lambda C}{2\mu(3\lambda+2\mu)} \\
C_{30} &= -\frac{\lambda C_{26}}{\lambda+2\mu} - C_{18} = \frac{-\lambda(5\lambda+2\mu)}{2(3\lambda+2\mu)^2} - \frac{4\mu^3A-3\lambda\mu(3\lambda+4\mu)B+3\lambda^2(\lambda+\mu)C}{\mu(3\lambda+2\mu)^3} \\
C_{31} &= -\frac{\lambda C_{27}}{\lambda+2\mu} - C_{19} = \frac{7\lambda+2\mu}{6(3\lambda+2\mu)} - \frac{B}{3(3\lambda+2\mu)} + \frac{\lambda C}{6\mu(3\lambda+2\mu)} \\
C_{32} &= -\frac{\lambda C_{28}}{\lambda+2\mu} - C_{20} = \frac{\lambda(7\lambda+2\mu)}{6(\lambda+2\mu)(3\lambda+2\mu)} - \frac{8\mu^3A+2\lambda\mu(\lambda-4\mu)B-\lambda^3C}{6\mu(\lambda+2\mu)^2(3\lambda+2\mu)}
\end{aligned}
\right.
\tag{4.36}
$$

From the above

$$\frac{1}{2}\,(v_{\alpha,\beta\gamma}+v_{\beta,\alpha\gamma}+w_{,\alpha\gamma}w_{,\beta} + w_{,\alpha}w_{,\beta\gamma}+v_{\nu,\alpha\gamma}v_{\nu,\beta}+v_{\nu,\alpha}v_{\nu,\beta\gamma})$$

$$= S_{\alpha\beta,\gamma} + \mathcal{O}(\frac{h^4}{D^5}\,\bar{\varepsilon}_2).$$

Permuting α,β,γ and taking the appropriate linear combination, we obtain the second result of (0.13).

Using (0.13) in (4.30), we obtain the second result of (0.19) where

$$(4.37)\left\{\begin{aligned}
C_{33} &= \frac{2\lambda\mu}{\lambda+2\mu} C_{25} + 2\mu C_{29} + \frac{2\mu B}{\lambda+2\mu} = \mu \\
C_{34} &= \frac{2\lambda\mu}{\lambda+2\mu} C_{26} + C_{11} + 2\mu C_{30} = -\frac{\lambda\mu}{3\lambda+2\mu} \\
C_{35} &= \frac{2\lambda\mu}{\lambda+2\mu} C_{27} + 2\mu C_{31} - \frac{\lambda\mu}{\lambda+2\mu} = \frac{\mu(-5\lambda^2+4\lambda\mu+8\mu^2)}{12(\lambda+\mu)(\lambda+2\mu)} - \frac{2\mu B}{3(\lambda+2\mu)} \\
C_{36} &= \frac{2\lambda\mu}{\lambda+2\mu} C_{28} + 2\mu C_{32} - \frac{\lambda^2\mu}{(\lambda+2\mu)^2} = \\
&= \frac{\lambda\mu(5\lambda+4\mu)}{12(\lambda+\mu)(\lambda+2\mu)} - \frac{8\mu^3A+6\lambda^2\mu B-\lambda^3C}{3(\lambda+2\mu)^3} \\
C_{37} &= C_{14} - 2\mu C_{21} = \frac{4\lambda\mu}{3\lambda+2\mu} \\
C_{38} &= C - 2\mu C_{22} = -4\mu
\end{aligned}\right.$$

Using (0.13) in (4.27), we obtain the second result of (0.16). Using (0.13) in (4.28), we obtain the second result of (0.18) where

$$(4.38)\left\{\begin{aligned}
C_{39} &= -\frac{\lambda}{\lambda+2\mu} C_{25} - \frac{B}{\lambda+2\mu} = \frac{\lambda}{3\lambda+2\mu} - \frac{B}{3\lambda+2\mu} + \frac{\lambda C}{2\mu(3\lambda+2\mu)} \\
C_{40} &= C_6 - \frac{\lambda}{\lambda+2\mu} C_{26} = \frac{-\lambda^2}{(3\lambda+2\mu)^2} - \frac{4\mu^3A-3\lambda\mu(3\lambda+4\mu)B+3\lambda^2(\lambda+\mu)C}{\mu(3\lambda+2\mu)^3} \\
C_{41} &= -\frac{\lambda}{\lambda+2\mu} C_{27} - \frac{\mu}{\lambda+2\mu} \\
&= \frac{7\lambda^3-42\lambda^2\mu-96\lambda\mu^2-48\mu^3}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} + \frac{2\lambda B}{3(\lambda+2\mu)(3\lambda+2\mu)} + \frac{\lambda C}{6\mu(3\lambda+2\mu)} \\
C_{42} &= \frac{-\lambda}{\lambda+2\mu} C_{28} - \frac{\lambda\mu}{(\lambda+2\mu)^2} \\
&= -\frac{\lambda(23\lambda^2+46\lambda\mu+24\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} + \frac{8\lambda\mu^3A+8\lambda\mu(\lambda^2+\lambda\mu+\mu^2)B-\lambda^4C}{3\mu(\lambda+2\mu)^3(3\lambda+2\mu)}
\end{aligned}\right.$$

From the second result of (0.18) and (4.2), we obtain the second result of (0.15) where

$$(4.39)\quad \begin{cases} C_{43} = C_{40} - \dfrac{\lambda^2}{2(3\lambda+2\mu)^2} = -\dfrac{3\lambda^2}{2(3\lambda+2\mu)^2} \\ \qquad\qquad - \dfrac{4\mu^3 A - 3\lambda\mu(3\lambda+4\mu)B + 3\lambda^2(\lambda+\mu)C}{\mu(3\lambda+2\mu)^3} \\ C_{44} = \dfrac{1}{3} C_8 - \dfrac{\lambda^2}{6(\lambda+2\mu)^2} = \dfrac{\lambda(\lambda^2+10\lambda\mu+8\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)^2} - \dfrac{4\mu^2 A + \lambda(\lambda-4\mu)B + \lambda^2 C}{3(\lambda+2\mu)^3} \\ C_{45} = \dfrac{3\lambda+4\mu}{6(\lambda+2\mu)} \\ C_{46} = -\dfrac{2(\lambda+\mu)}{\lambda+2\mu} \end{cases}$$

We have shown that

$$(4.40)\quad S_{\alpha\beta} = \frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha} + w_{,\alpha}w_{,\beta} + v_{\gamma,\alpha}v_{\gamma,\beta}\right) + \mathcal{O}\left(\frac{h^4}{D^4}\bar{\varepsilon}_2\right) .$$

Then $S_{\alpha\beta} = S_{\beta\alpha}$ and

$$S_{\alpha\beta,\gamma} = \frac{1}{2}\left(v_{\alpha,\beta\gamma} + v_{\beta,\alpha\gamma} + w_{,\alpha\gamma}w_{,\beta} + w_{,\alpha}w_{,\beta\gamma} + v_{\nu,\alpha\gamma}v_{\nu,\beta} + v_{\nu,\alpha}v_{\nu,\beta\gamma}\right) + \mathcal{O}\left(\frac{h^4}{D^5}\bar{\varepsilon}_2\right).$$

Permuting α,β,γ and taking the appropriate linear combination we have

$$v_{\alpha,\beta\gamma} + w_{,\alpha}w_{,\beta\gamma} + v_{\nu,\alpha}v_{\nu,\beta\gamma} = S_{\alpha\beta,\gamma} + S_{\alpha\gamma,\beta} - S_{\gamma\beta,\alpha} + \mathcal{O}\left(\frac{h^4}{D^5}\bar{\varepsilon}_2\right)$$

so that

$$v_{\alpha,\beta\gamma\nu} + w_{,\alpha\nu}w_{,\beta\gamma} + w_{,\alpha}w_{,\beta\gamma\nu} + v_{\omega,\alpha\nu}v_{\omega,\beta\gamma} + v_{\omega,\alpha}v_{\omega,\beta\gamma\nu}$$
$$= S_{\alpha\beta,\gamma\nu} + S_{\alpha\gamma,\beta\nu} - S_{\gamma\beta,\alpha\nu} + \mathcal{O}\left(\frac{h^4}{D^6}\bar{\varepsilon}_2\right) .$$

Interchanging γ and ν and subtracting gives

$$w_{,\alpha\nu}w_{,\beta\gamma} - w_{,\alpha\gamma}w_{,\beta\nu} + v_{\omega,\alpha\nu}v_{\omega,\beta\gamma} - v_{\omega,\alpha\gamma}v_{\omega,\beta\nu}$$

$$= S_{\alpha\gamma,\beta\nu} - S_{\alpha\nu,\beta\gamma} - S_{\gamma\beta,\alpha\nu} + S_{\nu\beta,\alpha\gamma} + \mathcal{O}\left(\frac{h^4}{D^6}\bar{\varepsilon}_2\right) .$$

Setting $\alpha = \gamma$, $\beta = \nu$ and summing, this becomes

$$(4.41)\quad \Delta S_{\gamma\gamma} - S_{\gamma\nu,\gamma\nu} = \frac{1}{2}\left[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2 + v_{\omega,\gamma\nu}v_{\omega,\gamma\nu} - \Delta v_\omega \Delta v_\omega\right] + \mathcal{O}\left(\frac{h^4}{D^6}\bar{\varepsilon}_2\right) .$$

From (0.13) we obtain

$$(4.42)\quad S_{\gamma\gamma} = \frac{\lambda\ +2\mu}{3\mu+2\mu}\Delta\psi + C_{25}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{26}(\Delta\psi)^2$$

$$+ h^2\left[C_{27}w_{,\gamma\nu}w_{,\gamma\nu} + C_{28}(\Delta w)^2\right] + \mathcal{O}\left(\frac{h^4}{D^4}\bar{\varepsilon}_2\right)$$

and

$$(4.43)\quad S_{\gamma\nu,\gamma\nu} = -\frac{\lambda}{3\lambda+2\mu}\Delta^2\psi + C_{51}\psi_{,\gamma\nu\omega}\psi_{,\gamma\nu\omega} + 2C_{29}\psi_{,\gamma\nu}\Delta\psi_{,\gamma\nu} + C_{52}\Delta\psi_{,\gamma}\Delta\psi_{,\gamma}$$

$$- C_{21}T_{\gamma\nu}\Delta\psi_{,\gamma\nu} + h^2\left[C_{53}w_{,\gamma\nu\omega}w_{,\gamma\nu\omega} + C_{54}w_{,\gamma\nu}\Delta w_{,\gamma\nu} + C_{55}\Delta w_{,\gamma}\Delta w_{,\gamma}\right]$$

$$+ \frac{3\lambda+2\mu}{2(\lambda+\mu)}C_{30}\Delta\psi\left[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2\right] + C_{56}T_{\gamma\nu}w_{,\gamma\nu}\Delta w + \mathcal{O}\left(\frac{h^4}{D^6}\bar{\varepsilon}_2\right)$$

where

$$(4.44)\left\{\begin{aligned}
C_{51} &= 2C_{29} - C_{22} = C_{25}\\
C_{52} &= 2C_{30} + C_{22} = \frac{13\lambda^2+22\lambda\mu+8\mu^2}{(3\lambda+2\mu)^2}\\
&\quad - \frac{16\mu^3A - 12\lambda\mu(3\lambda+4\mu)B - (15\lambda^3+42\lambda^2\mu+36\lambda\mu^2+8\mu^3)C}{2\mu(3\lambda+2\mu)^3}\\
C_{53} &= 2C_{31} - C_{24} - \frac{\lambda}{12(\lambda+\mu)}\\
&= -\frac{(5\lambda+6\mu)(\lambda+2\mu)}{12(\lambda+\mu)(3\lambda+2\mu)} - \frac{2B}{3(3\lambda+2\mu)} - \frac{\lambda+2\mu}{6\mu(3\lambda+2\mu)}C
\end{aligned}\right.$$

$$\left\{\begin{aligned}
C_{54} &= 2C_{31} - C_{23} - 2C_{24} - \frac{\lambda}{12(\lambda+\mu)} \\
&= -\frac{53\lambda^3+166\lambda^2\mu+176\lambda\mu^2+64\mu^3}{12(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} - \frac{8(\lambda+\mu)B}{3(\lambda+2\mu)(3\lambda+2\mu)} - \frac{2(\lambda+\mu)}{3\mu(3\lambda+2\mu)}C \\
C_{55} &= 2C_{32} - 2C_{23} - C_{24} + \frac{\lambda}{12(\lambda+\mu)} = -\frac{35\lambda^3+126\lambda^2\mu+132\lambda\mu^2+40\mu^3}{12(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} \\
&\quad - \frac{16\mu^3A+4\mu(7\lambda^2+12\lambda\mu+8\mu^2)B+(\lambda^3+14\lambda^2\mu+20\lambda\mu^2+8\mu^3)C}{6\mu(\lambda+2\mu)^2(3\lambda+2\mu)} \\
C_{56} &= \frac{3(\lambda+2\mu)}{2(\lambda+\mu)}\left[2C_{32} - C_{23} + \frac{\lambda}{12(\lambda+\mu)}\right] \\
&= \frac{\lambda^2(13\lambda+14\mu)}{8(\lambda+\mu)^2(3\lambda+2\mu)} - \frac{8\mu^3A+8\mu(\lambda^2+\lambda\mu+\mu^2)B-\lambda^3C}{2\mu(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)}
\end{aligned}\right.$$

From (4.41), (4.42), (4.43), (4.13), (4.14), and (4.20) we obtain the first result of (0.22) where

$$\left\{\begin{aligned}
C_{57} &= \frac{3\lambda+2\mu}{2(\lambda+\mu)}\left(\frac{1}{2} - C_{25}\right) = \frac{5\lambda+6\mu}{4(\lambda+\mu)} + \frac{B}{\lambda+\mu} + \frac{\lambda+2\mu}{4\mu(\lambda+\mu)}C \\
C_{58} &= \frac{3\lambda+2\mu}{\lambda+\mu}(C_{29}-C_{25}) = \frac{7\lambda+6\mu}{2(\lambda+\mu)} + \frac{B}{\lambda+\mu} + \frac{C}{\mu} \\
C_{59} &= \frac{3\lambda+2\mu}{2(\lambda+\mu)}\left[\frac{-5\lambda^2+4\mu^2}{2(3\lambda+2\mu)^2} - 2C_{26} + C_{52}\right] = \frac{17\lambda^2+36\lambda\mu+20\mu^2}{4(\lambda+\mu)(3\lambda+2\mu)} \\
&\quad + \frac{16\mu^3A+4\mu(9\lambda^2+12\lambda\mu+8\mu^2)B+(3\lambda^3+18\lambda^2\mu+20\lambda\mu^2+8\mu^3)C}{4\mu(\lambda+\mu)(3\lambda+2\mu)^2} \\
C_{60} &= -\frac{3\lambda+2\mu}{2(\lambda+\mu)}C_{21} = \frac{\lambda}{\lambda+\mu} - \frac{B}{\lambda+\mu} + \frac{\lambda C}{2\mu(\lambda+\mu)} \\
C_{61} &= -\frac{3\lambda+2\mu}{\lambda+\mu}(C_{26} - C_{30}) = -\frac{\lambda(7\lambda+6\mu)}{2(\lambda+\mu)(3\lambda+2\mu)} \\
&\quad + \frac{4\mu^3A+\mu(9\lambda^2+12\lambda\mu+8\mu^2)B-\lambda(6\lambda^2+9\lambda\mu+4\mu^2)C}{\mu(\lambda+\mu)(3\lambda+2\mu)^2} \\
C_{62} &= \frac{3\lambda+2\mu}{2(\lambda+\mu)}\left[C_{56} - \frac{3(\lambda+2\mu)}{\lambda+\mu}C_{28}\right] \\
&= -\frac{\lambda(5\lambda-2\mu)}{8(\lambda+\mu)^2} + \frac{8\mu^3A+8\mu(\lambda^2+\lambda\mu+\mu^2)B-\lambda^3C}{4\mu(\lambda+\mu)^2(\lambda+2\mu)}
\end{aligned}\right. \tag{4.45}$$

$$
\left\{
\begin{aligned}
C_{63} &= \frac{3\lambda+2\mu}{2(\lambda+\mu)}\,(C_{53}-2C_{27}) = \frac{\lambda+6\mu}{12(\lambda+\mu)} + \frac{B}{3(\lambda+\mu)} + \frac{\lambda+2\mu}{12\mu(\lambda+\mu)}\,C \\
C_{64} &= \frac{3\lambda+2\mu}{2(\lambda+\mu)}\,(C_{54}-2C_{27}) = -\frac{23\lambda^2+38\lambda\mu+8\mu^2}{12(\lambda+\mu)(\lambda+2\mu)} - \frac{2\lambda B}{3(\lambda+\mu)(\lambda+2\mu)} \\
&\qquad - \frac{\lambda C}{6\mu(\lambda+\mu)} \\
C_{65} &= \frac{3\lambda+2\mu}{2(\lambda+\mu)}\,(C_{55}-2C_{28}) = -\frac{29\lambda^2+44\lambda\mu+20\mu^2}{12(\lambda+\mu)(\lambda+2\mu)} \\
&\quad + \frac{16\mu^3A+4\lambda\mu(\lambda-4\mu)B-(5\lambda^3+14\lambda^2\mu+20\lambda\mu^2+8\mu^3)C}{12\mu(\lambda+\mu)(\lambda+2\mu)^2}
\end{aligned}
\right.
$$

From line 1 of (0.16) and (0.18), we have

$$
(4.46)\quad e_{kk1} = x_3\left\{-\frac{2\mu}{\lambda+2\mu}\,(\Delta w + w_{,\gamma}\Delta\psi_{,\gamma} + \frac{1}{2}\,w_{,\gamma}w_{,\gamma}\Delta w) + (C_{66} + C_{67}\,\frac{x_3^2}{h^2})\,T_{\gamma\nu}w_{,\gamma\nu} + C_{68}\Delta\psi\;\Delta w\right\} + \mathcal{O}(\frac{h^4}{D^4}\,\varepsilon_1)
$$

where

$$
(4.47)\left\{
\begin{aligned}
C_{66} &= C_1 - 3 = \frac{-5\mu+2B}{\lambda+2\mu} \\
C_{67} &= C_2 + \frac{3\lambda+4\mu}{4(\lambda+\mu)} = \frac{\mu}{2(\lambda+\mu)} \\
C_{68} &= C_3 + \frac{\lambda}{3\lambda+2\mu} = \frac{2\lambda\mu}{(3\lambda+2\mu)(\lambda+2\mu)} + \frac{8\mu^2A-12\lambda\mu B+2\lambda^2C}{(3\lambda+2\mu)(\lambda+2\mu)^2}
\end{aligned}
\right.
$$

From (1.4), line 1 of (0.16), and (4.46) we obtain line 1 of (0.19) where

$$
(4.48)\left\{
\begin{aligned}
C_{69} &= \lambda C_{66} - 2B = -\frac{\mu(5\lambda+4B)}{\lambda+2\mu} \\
C_{70} &= \lambda C_{68} + \frac{8\mu(\lambda B-\mu A)}{(\lambda+2\mu)(3\lambda+2\mu)} = \frac{2\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)} \\
&\qquad - \frac{16\mu^3A+4\lambda\mu(\lambda-4\mu)B-2\lambda^3C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\
C_{71} &= \frac{2\lambda\mu-4\mu B+2\lambda C}{3\lambda+2\mu}
\end{aligned}
\right.
$$

From (1.6)

$$
\begin{aligned}
p_{\alpha\beta 1} = {} & \delta_{\alpha\beta}x_3[2\mu T_{\gamma\nu}w_{,\gamma\nu} + \frac{4\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\ \Delta\psi\ \Delta w] \\
& + x_3[-4\mu(w_{,\alpha\gamma}T_{\gamma\beta} + w_{,\beta\gamma}T_{\gamma\alpha}) - \frac{4\lambda\mu}{\lambda+2\mu}\ T_{\alpha\beta}\Delta w + \frac{4\lambda\mu}{3\lambda+2\mu}\ w_{,\alpha\beta}\Delta\psi] \\
& + \quad (\frac{\mu h^4}{D^4}\ \varepsilon_1)\ .
\end{aligned}
$$

From this and line 1 of (0.19) we obtain

$$
\begin{aligned}
t_{\alpha\beta 1} + p_{\alpha\beta 1} = {} & \delta_{\alpha\beta}x_3\Big\{- \frac{2\lambda\mu}{\lambda+2\mu}\ \Delta w(1+ \frac{1}{2}\ w_{,\gamma}w_{,\gamma}) - \frac{4\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\ w_{,\gamma}\Delta\psi_{,\gamma} \\
& \qquad + (C_{72} + \lambda C_{67}\ \frac{x_3^2}{h^2})T_{\gamma\nu}w_{,\gamma\nu} + C_{73}\Delta\psi\ \Delta w\Big\} \\
& + x_3\Big\{-2\mu w_{,\alpha\beta} + [\ \frac{\mu(3\lambda+4\mu)}{3(\lambda+2\mu)}\ x_3^2 - \frac{4\mu(\lambda+\mu)}{\lambda\ +2\mu}\ h^2]\Delta w_{,\alpha\beta} + C_{74}w_{,\alpha\beta}\Delta\psi \\
& - \mu w_{,\gamma}w_{,\gamma}w_{,\alpha\beta} + 2\mu w_{,\gamma}(T_{\alpha\beta,\alpha} + T_{\alpha\gamma,\beta} - T_{\alpha\beta,\gamma}) \\
& - \frac{2\lambda\mu}{3\lambda+2\mu}\ (w_{,\alpha}\Delta\psi_{,\beta} + w_{,\beta}\Delta\psi_{,\alpha}) - \frac{4\mu(\lambda+B)}{\lambda+2\mu}\ T_{\alpha\beta}\Delta w \\
& - (4\mu+C)(w_{,\alpha\gamma}T_{\gamma\beta} + w_{,\beta\gamma}T_{\gamma\alpha})\Big\} + \quad (\frac{\mu h^4}{D^4}\ \varepsilon_1)
\end{aligned}
$$

where

$$
(4.49)\left\{
\begin{aligned}
C_{72} &= C_{69} + 2\mu = \frac{-3\lambda\mu+4\mu^2-4\mu B}{\lambda+2\mu} \\
C_{73} &= C_{70} + \frac{4\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)} = \frac{6\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)} \\
&\qquad - \frac{16\mu^3A+4\lambda\mu(\lambda-4\mu)B-2\lambda^3C}{(\lambda+2\mu)^2(3\lambda\ +\ 2\mu)} \\
C_{74} &= C_{71} + \frac{4\lambda\mu}{3\lambda+2\mu} = \frac{6\lambda\mu-4\mu B+2\lambda C}{3\lambda+2\mu}\ .
\end{aligned}
\right.
$$

Thus

$$
\begin{aligned}
(4.50)\quad (t_{\alpha\beta 1}+p_{\alpha\beta 1})_{,\beta} = x_3 \Bigg\{ & -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(1+\frac{1}{2}w_{,\gamma}w_{,\gamma}\right)\Delta w_{,\alpha} \\
& -\frac{2\lambda\mu}{\lambda+2\mu}(w_{,\gamma}w_{,\gamma\alpha}\Delta w + w_{,\gamma}\Delta\psi_{,\gamma\alpha}) + C_{75}w_{,\alpha\gamma}\Delta\psi_{,\gamma} + \left(C_{76}+C_{77}\frac{x_3^2}{h^2}\right)T_{\gamma\nu,\alpha}w_{,\gamma\nu} \\
& + \left(C_{78}+C_{77}\frac{x_3^2}{h^2}\right)T_{\gamma\nu}w_{,\gamma\nu\alpha} + C_{79}\,\Delta w\,\Delta\psi_{,\alpha} + C_{80}\,\Delta\psi\,\Delta w_{,\alpha} \\
& + 2\mu(w_{,\gamma}\Delta T_{\alpha\gamma} - w_{,\gamma}w_{,\gamma\nu}w_{,\alpha\nu}) + C_{81}T_{\alpha\gamma}\Delta w_{,\gamma} - (4\mu+C)w_{,\gamma\nu}T_{\gamma\alpha,\nu} \\
& - \frac{\lambda\mu}{2(\lambda+\mu)}w_{,\alpha}[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2]\Bigg\} + \mathcal{O}\left(\frac{\mu h^4}{D^5}\varepsilon_1\right)
\end{aligned}
$$

where

$$
(4.51)\quad \begin{cases}
C_{75} = C_{74} - \dfrac{2\lambda\mu}{\lambda+2\mu} = \dfrac{8\lambda\mu^2}{(\lambda+2\mu)(3\lambda+2\mu)} + \dfrac{2\lambda C-4\mu B}{3\lambda+2\mu} \\[2ex]
C_{76} = C_{72} - 4\mu = -\dfrac{7\lambda\mu+4\mu^2+4\mu B}{\lambda+2\mu} \\[2ex]
C_{77} = \lambda C_{67} + \dfrac{\mu(3\lambda+4\mu)}{2(\lambda+\mu)} = 2\mu \\[2ex]
C_{78} = C_{72} - 10\mu - C = -\dfrac{13\lambda\mu+16\mu^2+4\mu B}{\lambda+2\mu} - C \\[2ex]
C_{79} = C_{73} - \dfrac{2\lambda\mu}{3\lambda+2\mu} = \dfrac{4\lambda\mu(\lambda-\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} - \dfrac{16\mu^3A+4\lambda\mu(\lambda-4\mu)B-2\lambda^3C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\[2ex]
C_{80} = C_{73} + C_{74} = \dfrac{12\lambda\mu(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\[1ex]
\qquad\qquad - \dfrac{16\mu^3A+8\mu(\lambda^2+2\mu^2)B-4\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\[2ex]
C_{81} = -\dfrac{8\mu(\lambda+\mu)}{\lambda+2\mu} - \dfrac{4\mu B}{\lambda+2\mu} - C
\end{cases}
$$

Since $(t_{\alpha 31}+p_{\alpha 31})_{,3} = -(t_{\alpha\beta 1}+p_{\alpha\beta 1})_{,\beta}$, (4.50) gives

$$
\begin{aligned}
(4.52)\quad t_{\alpha 31}+p_{\alpha 31} = \frac{1}{2}(x_3^2-h^2)\Bigg\{ & \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(1+\frac{1}{2}w_{,\gamma}w_{,\gamma}\right)\Delta w_{,\alpha} \\
& + \frac{2\lambda\mu}{\lambda+2\mu}(w_{,\gamma}w_{,\gamma\alpha}\Delta w + w_{,\gamma}\Delta\psi_{,\gamma\alpha}) - C_{75}w_{,\alpha\gamma}\Delta\psi_{,\gamma} \\
& - \left(C_{82}+\frac{1}{2}C_{77}\frac{x_3^2}{h^2}\right)T_{\gamma\nu,\alpha}w_{,\gamma\nu} - \left(C_{83}+\frac{1}{2}C_{77}\frac{x_3^2}{h^2}\right)T_{\gamma\nu}w_{,\gamma\nu\alpha}
\end{aligned}
$$

$$- C_{79}\Delta w\Delta\psi_{,\alpha} - C_{80}\Delta\psi\,\Delta w_{,\alpha} - 2\mu(w_{,\gamma}\Delta T_{\alpha\gamma} - w_{,\gamma}w_{,\gamma\nu}w_{,\alpha\nu})$$

$$- C_{81}T_{\alpha\gamma}\Delta w_{,\gamma} + (4\mu+C)w_{,\gamma\nu}T_{\gamma\alpha,\nu} + \frac{\lambda\mu}{2(\lambda+\mu)}\, w_{,\alpha}[w_{,\gamma\nu}w_{,\gamma\nu} - (\Delta w)^2]\Big\}$$

$$+ \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\varepsilon_1\right)$$

where

$$(4.53)\begin{cases} C_{82} = C_{76} + \frac{1}{2}\,C_{77} = -\dfrac{6\lambda\mu+2\mu^2+4\mu B}{\lambda+2\mu} \\[2ex] C_{83} = C_{78} + \frac{1}{2}\,C_{77} = -\dfrac{12\lambda\mu+14\mu^2+4\mu B}{\lambda+2\mu} - C. \end{cases}$$

From (1.6)

$$p_{\alpha 31} = \frac{1}{2}\,(x_3^2-h^2)\left\{\frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\,T_{\alpha\gamma}\Delta w_{,\gamma} - \frac{8\lambda\mu(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)}\,\Delta\psi\Delta w_{,\alpha}\right\} + \mathcal{O}\left(\frac{\mu h^5}{D^5}\varepsilon_1\right).$$

From this and (4.52) we obtain line 1 of (0.20) where

$$(4.54)\begin{cases} C_{84} = C_{80} - \dfrac{8\lambda\mu(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\[2ex] \quad = \dfrac{4\lambda\mu(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} - \dfrac{16\mu^3A+8\mu(\lambda^2+2\mu^2)B-4\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\[2ex] C_{85} = -C_{81} - \dfrac{8\mu(\lambda+\mu)}{\lambda+2\mu} = \dfrac{4\mu B}{\lambda+2\mu} + C \end{cases}$$

From (1.3) and (1.2) we obtain

$$t_{\alpha 31} = 2\mu e_{\alpha 31} + 2Bs_{1e}e_{\alpha 31} + C(e_{\alpha\gamma 2}e_{\gamma 31} + e_{\alpha 31}e_{332}) + \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\varepsilon_1\right)$$

$$= 2\mu e_{\alpha 31} + (x_3^2-h^2)\left[\frac{2(\lambda+\mu)(2\mu B-\lambda C)}{(\lambda+2\mu)(3\lambda+2\mu)}\,\Delta\psi\Delta w_{,\alpha} + \frac{\lambda+\mu}{\lambda+2\mu}\,C\,T_{\alpha\gamma}\Delta w_{,\gamma}\right]$$

$$+ \mathcal{O}\left(\frac{\mu h^5}{D^5}\,\varepsilon_1\right).$$

From this and line 1 of (0.20) we obtain line 1 of (0.17) where

$$(4.55)\quad \left\{\begin{aligned} C_{86} &= -\frac{C_{84}}{2\mu} - \frac{2(\lambda+\mu)(2\mu B-\lambda C)}{\mu(\lambda+2\mu)(3\lambda+2\mu)} \\ &= -\frac{2\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} + \frac{8\mu^2 A-12\lambda\mu B+2\lambda^2 C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\ C_{87} &= \frac{C_{85}}{2\mu} - \frac{\lambda+\mu}{\mu(\lambda+2\mu)}\, C \;=\; \frac{2B}{\lambda+2\mu} - \frac{\lambda C}{2\mu(\lambda+2\mu)} \end{aligned}\right.$$

From the first result of (0.17) and (4.2) we obtain the first result of (0.14) where

$$C_{88} = \frac{\lambda(7\lambda+6\mu)}{6(\lambda+2\mu)(3\lambda+2\mu)} - \frac{1}{6}\, C_4 = \frac{\lambda}{6(\lambda+2\mu)}$$

$$C_{89} = \frac{\lambda}{3(3\lambda+2\mu)} - \frac{1}{6}\, C_4 - \frac{1}{6\mu}\, C_{75} = -\frac{\lambda(\lambda+4\mu)}{3(\lambda+2\mu)(3\lambda+2\mu)} + \frac{2\mu B-\lambda C}{3\mu(3\lambda+2\mu)}$$

$$C_{90} = -\frac{1}{10} - \frac{1}{20}\, C_2 = -\frac{5\lambda+6\mu}{80(\lambda+\mu)}$$

$$C_{91} = -\frac{1}{6\mu}\, C_{82} + \frac{1}{6} - \frac{1}{6}\, C_1 = \frac{4\lambda+3\mu+2B}{6(\lambda+2\mu)}$$

$$C_{92} = -\frac{1}{6\mu}\, C_{83} + \frac{1}{6} - \frac{1}{6}\, C_1 = \frac{10\lambda+15\mu+2B}{6(\lambda+2\mu)} + \frac{C}{6\mu}$$

$$\begin{aligned} C_{93} &= -\frac{1}{6\mu}\, C_{79} - \frac{1}{6}\, C_5 + \frac{\lambda^2}{3(\lambda+2\mu)(3\lambda+2\mu)} \\ &= \frac{\lambda(-\lambda+2\mu)}{3(\lambda+2\mu)(3\lambda+2\mu)} + \frac{4\mu^3 A+2\lambda\mu(\lambda-\mu)B-\lambda^2(\lambda+\mu)C}{3\mu(\lambda+2\mu)^2(3\lambda+2\mu)} \end{aligned}$$

$$\begin{aligned} C_{94} &= \frac{1}{3}\, C_{86} - \frac{1}{6}\, C_5 + \frac{\lambda^2}{6(\lambda+2\mu)(3\lambda+2\mu)} \\ &= -\frac{\lambda(3\lambda+4\mu)}{6(\lambda+2\mu)(3\lambda+2\mu)} + \frac{4\mu^2 A-6\lambda\mu B+\lambda^2 C}{3(\lambda+2\mu)^2(3\lambda+2\mu)} \end{aligned}$$

$$C_{95} = \frac{\lambda}{12(\lambda+\mu)} - \frac{1}{3}\, C_7 = \frac{7\lambda^2-8\mu^2}{24(\lambda+\mu)(\lambda+2\mu)} + \frac{B}{3(\lambda+2\mu)}$$

$$\begin{aligned} C_{96} &= -\frac{\lambda}{4(\lambda+\mu)} - C_{41} \\ &= -\frac{25\lambda^3+6\lambda^2\mu-72\lambda\mu^2-48\mu^3}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} - \frac{2\lambda B}{3(\lambda+2\mu)(3\lambda+2\mu)} - \frac{\lambda C}{6\mu(3\lambda+2\mu)} \end{aligned}$$

$$C_{97} = -\frac{\lambda}{12(\lambda+\mu)} - C_{44} = -\frac{\lambda(3\lambda^2+18\lambda\mu+16\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)^2} + \frac{4\mu^2A+\lambda(\lambda-4\mu)B+\lambda^2C}{3(\lambda+2\mu)^3}$$

$$C_{98} = \frac{\lambda}{4(\lambda+\mu)} - C_{42} = \frac{\lambda(41\lambda^2+94\lambda\mu+48\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} - \frac{8\lambda\mu^3A+8\lambda\mu(\lambda^2+\lambda\mu+\mu^2)B-\lambda^4C}{3\mu(\lambda+2\mu)^3(3\lambda+2\mu)}$$

From (4.52)

$$(4.56)\quad (t_{3\alpha 1}+p_{3\alpha 1})_{,\alpha} = \frac{1}{2}(x_3^2-h^2)\Big\{C_{101}w_{,\gamma}w_{,\gamma\nu}\Delta w_{,\nu} + C_{102}w_{,\gamma\nu}w_{,\gamma\nu}\Delta w + C_{103}w_{,\gamma}\Delta w_{,\gamma}\Delta w + C_{104}w_{,\gamma\nu}\Delta\psi_{,\gamma\nu} + C_{105}w_{,\gamma}w_{,\nu\omega}w_{,\gamma\nu\omega} + C_{106}\Delta^2 w + \frac{3\mu}{h^2}T_{\gamma\nu}w_{,\gamma\nu}w_{,\omega}w_{,\omega} + C_{107}\Delta\psi_{,\gamma}\Delta w_{,\gamma} - \Big(C_{108}+\mu\frac{x_3^2}{h^2}\Big)\Delta T_{\gamma\nu}w_{,\gamma\nu} + \Big(C_{109}-2\mu\frac{x_3^2}{h^2}\Big)T_{\gamma\nu,\omega}w_{,\gamma\nu\omega} + \Big(C_{110}-\mu\frac{x_3^2}{h^2}\Big)T_{\gamma\nu}\Delta w_{,\gamma\nu} + C_{111}(\Delta w)^3 + C_{112}\frac{1}{h^2}T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi + 2\mu w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma}\Big\} + \mathcal{O}\Big(\frac{\mu h^5}{D^6}\varepsilon_1\Big)$$

where

$$(4.57)\quad \left\{\begin{aligned}
C_{101} &= \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\\
C_{102} &= \frac{\lambda\mu(5\lambda+6\mu)}{2(\lambda+\mu)(\lambda+2\mu)} - \frac{3\lambda+2\mu}{4(\lambda+\mu)}C_{79}\\
&= \frac{\lambda\mu(3\lambda+8\mu)}{2(\lambda+\mu)(\lambda+2\mu)} + \frac{8\mu^3A+2\lambda\mu(\lambda-4\mu)B-\lambda^3C}{2(\lambda+\mu)(\lambda+2\mu)^2}\\
C_{103} &= -\frac{2\lambda\mu}{\lambda+2\mu}\\
C_{104} &= \frac{2\lambda\mu}{\lambda+2\mu} - C_{75} = \frac{2\lambda\mu(3\lambda-2\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} + \frac{4\mu B-2\lambda C}{3\lambda\ +\ 2\mu}\\
C_{105} &= \frac{2\mu(3\lambda+2\mu)}{\lambda+2\mu}\\
C_{106} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}
\end{aligned}\right.$$

$$\left\{\begin{array}{l}
C_{107} = -C_{80} - C_{79} - C_{75} = -\dfrac{16\lambda\mu(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} \\[2ex]
\qquad + \dfrac{32\mu^3 A + 16\mu(\lambda^2+2\mu^2)B - 8\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\[3ex]
C_{108} = C_{82} + 2\mu = \dfrac{-4\lambda\mu+2\mu^2-4\mu B}{\lambda+2\mu} \\[3ex]
C_{109} = -C_{82} - C_{83} + 4\mu + C = \dfrac{22\lambda\mu+24\mu^2+8\mu B}{\lambda+2\mu} + 2C \\[3ex]
C_{110} = -C_{81} - C_{83} = \dfrac{20\lambda\mu+22\mu^2+8\mu B}{\lambda+2\mu} + 2C \\[3ex]
C_{111} = \dfrac{3\lambda+2\mu}{4(\lambda+\mu)} C_{79} - \dfrac{\lambda\mu}{2(\lambda+\mu)} = \dfrac{\lambda\mu(\lambda-4\mu)}{2(\lambda+\mu)(\lambda+2\mu)} \\[2ex]
\qquad - \dfrac{8\mu^3 A + 2\lambda\mu(\lambda-4\mu)B - \lambda^3 C}{2(\lambda+\mu)(\lambda+2\mu)^2} \\[3ex]
C_{112} = -\dfrac{3(\lambda+2\mu)}{2(\lambda+\mu)} C_{80} = -\dfrac{18\lambda\mu}{3\lambda+2\mu} \\[2ex]
\qquad + \dfrac{24\mu^3 A + 12\mu(\lambda^2+2\mu^2)B - 6\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)}
\end{array}\right.$$

From (1.6)

$$(4.58) \quad p_{331} = x_3(x_3^2-h^2)\left\{\frac{2\lambda\mu}{\lambda+2\mu}\Delta w\left[w_{,\gamma\nu}w_{,\gamma\nu} + \frac{\lambda}{\lambda+2\mu}(\Delta w)^2\right] + \frac{2\lambda\mu}{3\lambda+2\mu}\frac{1}{h^2}T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi\right\} - W_o + \mathcal{O}\left(\frac{\mu h^6}{D^6}\varepsilon_1\right)$$

From (1.1)

$$(4.59) \quad s_{2o} = 2e_{\gamma\nu 1}e_{\gamma\nu 2} + 2e_{331}e_{332} + \mathcal{O}\left(\frac{h^6}{D^6}\varepsilon_1\right)$$

$$= x_3\left\{\frac{4\lambda\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\Delta\psi\,\Delta w + C_{113}\psi_{,\gamma\nu}\psi_{,\gamma\nu}\Delta w + C_{114}(\Delta\psi)^2\Delta w\right.$$

$$+ (C_{115}x_3^2 + C_{116}h^2)w_{,\gamma\nu}w_{,\gamma\nu}\Delta w + (C_{117}x_3^2 + C_{118}h^2)(\Delta w)^3$$

$$- 2T_{\gamma\nu}w_{,\gamma\nu} + \left(C_{119}\frac{x_3^2}{h^2} + C_{120}\right)T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi + 2C_{22}T_{\gamma\nu}T_{\nu\omega}w_{,\omega\gamma}$$

$$- \frac{\lambda}{3\lambda+2\mu}\left(x_3^2 - \frac{1}{3}h^2\right)\Delta\psi_{,\gamma\nu}w_{,\gamma\nu} - (x_3^2-2C_{24}h^2)w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma}$$

$$+ \left[\frac{3\lambda+4\mu}{3(\lambda+2\mu)}x_3^2 - \frac{4(\lambda+\mu)}{\lambda+2\mu}h^2\right]\Delta w_{,\gamma\nu}T_{\gamma\nu} - T_{\gamma\nu}w_{,\gamma\nu}w_{,\omega}w_{,\omega}$$

$$+ 4w_{,\omega}T_{\gamma\nu}T_{\omega\gamma,\nu} - 2\psi_{,\gamma\nu}\psi_{,\gamma\nu\omega}w_{,\omega} - \frac{4\lambda}{3\lambda+2\mu}T_{\gamma\nu}\Delta\psi_{,\gamma}w_{,\nu}$$

$$+ \frac{2\lambda\mu}{(\lambda+2\mu)(3\lambda+2\mu)}w_{,\gamma}w_{,\gamma}\Delta w\Delta\psi + \frac{2\lambda(\lambda+4\mu)}{(\lambda+2\mu)(3\lambda+2\mu)}\Delta\psi\,\Delta\psi_{,\gamma}w_{,\gamma}\Big\}$$

$$+ \mathcal{O}\left(\frac{h^6}{D^6}\varepsilon_1\right)$$

where

$$\left\{\begin{aligned}
C_{113} &= \frac{2\lambda}{\lambda+2\mu}C_{39} - 2C_{29} = \frac{-3\lambda^2-12\lambda\mu-4\mu^2+4\mu B-2\lambda C}{(\lambda+2\mu)(3\lambda+2\mu)}\\
C_{114} &= \frac{2\lambda}{\lambda+2\mu}C_{40} - \frac{2\lambda}{3\lambda+2\mu}C_3 - 2C_{30} - \frac{2\lambda^2}{(3\lambda+2\mu)^2} = \frac{\lambda}{3\lambda+2\mu}\\
&\quad - \frac{16\mu^2(3\lambda-2\mu)(\lambda+\mu)A-12\lambda\mu(3\lambda^2-6\lambda\mu-8\mu^2)B-4\lambda^2\mu(7\lambda+6\mu)C}{(\lambda+2\mu)^2(3\lambda+2\mu)^3}\\
C_{115} &= \frac{2\lambda}{\lambda+2\mu}C_7 + \frac{\lambda}{\lambda+2\mu} = \frac{\lambda(-\lambda^2+16\lambda\mu+16\mu^2)}{4(\lambda+\mu)(\lambda+2\mu)^2} - \frac{2\lambda B}{(\lambda+2\mu)^2}\\
C_{116} &= \frac{2\lambda}{\lambda+2\mu}C_{41} + 2C_{23} - 2C_{31} = \frac{15\lambda^4-58\lambda^3\mu-216\lambda^2\mu^2-176\lambda\mu^3-32\mu^4}{12(\lambda+\mu)(\lambda+2\mu)^2(3\lambda+2\mu)}\\
&\quad + \frac{2B(9\lambda^2+20\lambda\mu+12\mu^2)}{3(\lambda+2\mu)^2(3\lambda+2\mu)} - \frac{2\lambda C}{3(\lambda+2\mu)(3\lambda+2\mu)}\\
C_{117} &= \frac{2\lambda}{\lambda+2\mu}C_8 = \frac{\lambda^2(5\lambda+4\mu)}{4(\lambda+\mu)(\lambda+2\mu)^2} - \frac{8\lambda\mu^2A+2\lambda^2(\lambda-4\mu)B+2\lambda^3C}{(\lambda+2\mu)^4}\\
C_{118} &= \frac{2\lambda}{\lambda+2\mu}C_{42} - 2C_{32} = -\frac{\lambda(51\lambda^3+138\lambda^2\mu+104\lambda\mu^2+16\mu^3)}{12(\lambda+\mu)(\lambda+2\mu)^2(3\lambda+2\mu)}\\
&\quad + \frac{(8\mu^3A-\lambda^3C)(3\lambda^2+4\lambda\mu+4\mu^2)+2\lambda\mu B(9\lambda^3+8\lambda^2\mu-4\lambda\mu^2-16\mu^3)}{3\mu(\lambda+2\mu)^4(3\lambda+2\mu)}\\
C_{119} &= -\frac{2\lambda}{3\lambda+2\mu}C_2 - \frac{\lambda(3\lambda+4\mu)}{2(\lambda+\mu)(3\lambda+2\mu)} = -\frac{\lambda\mu}{(\lambda+\mu)(3\lambda+2\mu)}\\
C_{120} &= -\frac{2\lambda}{3\lambda+2\mu}C_1 + 2C_{21} + \frac{8\lambda}{3\lambda+2\mu} = \frac{2\lambda(-\lambda+3\mu)+8\mu B}{(\lambda+2\mu)(3\lambda+2\mu)} - \frac{2\lambda C}{\mu(3\lambda+2\mu)}
\end{aligned}\right. \tag{4.60}$$

Also from (1.1)

$$(4.61)\quad s_{3o} = 3x_3 w_{,\gamma\nu}\left(\frac{2\lambda}{3\lambda+2\mu} T_{\gamma\nu}\Delta\psi - T_{\gamma\omega}T_{\omega\nu}\right) - \frac{6\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)^2}\, x_3(\Delta\psi)^2\Delta w$$

$$- x_3^3 w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma} + \frac{\lambda^3}{(\lambda+2\mu)^3}\, x_3^3(\Delta w)^3 + \mathcal{O}\left(\frac{h^6}{D^6}\,\varepsilon_1\right) .$$

From the second results of (0.16) and (0.18)

$$(4.62)\quad e_{kk2} = \frac{2\mu}{3\lambda+2\mu}\,\Delta\psi + C_{121}\psi_{,\gamma\nu}\psi_{,\gamma\nu} + C_{122}(\Delta\psi)^2$$

$$+ (C_9 x_3^2 + C_{123}h^2)w_{,\gamma\nu}w_{,\gamma\nu} + (C_{10}x_3^2 + C_{124}h^2)(\Delta w)^2 + \mathcal{O}\left(\frac{h^4}{D^4}\,\bar{\varepsilon}_2\right)$$

where

$$(4.63)\left\{\begin{aligned}
C_{121} &= \frac{2\mu}{\lambda+2\mu}\,C_{25} - \frac{B}{\lambda+2\mu} = -\frac{2\mu+3B+C}{3\lambda+2\mu}\\
C_{122} &= \frac{2\mu}{\lambda+2\mu}\,C_{26} + C_6 = \frac{2\ \lambda\mu}{(3\lambda+2\mu)^2}\\
&\quad - \frac{12\mu^2A-(9\lambda^2+12\lambda\mu-8\mu^2)B-\lambda(3\lambda+4\mu)C}{(3\lambda+2\mu)^3}\\
C_{123} &= \frac{2\mu}{\lambda+2\mu}\,C_{27} - \frac{\mu}{\lambda+2\mu} = -\frac{\mu(43\lambda^2+90\lambda\mu+48\mu^2)}{12(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)}\\
&\quad - \frac{4\mu B}{3(\lambda+2\mu)(3\lambda+2\mu)} - \frac{C}{3(3\lambda+2\mu)}\\
C_{124} &= \frac{2\mu}{\lambda+2\mu}\,C_{28} - \frac{\lambda\mu}{(\lambda+2\mu)^2}\\
&= -\frac{\lambda\mu(13\lambda+14\mu)}{12(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} - \frac{16\mu^3A+16\mu(\lambda^2+\lambda\mu+\mu^2)B-2\lambda^3C}{3(\lambda+2\mu)^3(3\lambda+2\mu)}
\end{aligned}\right.$$

From (1.4), (4.59), (4.61), (4.62), and (4.19)

$$
\begin{aligned}
(4.64)\quad W_o = x_3\Big\{ & C_{125}\psi_{,\gamma\nu}\psi_{,\gamma\nu}\Delta w + C_{126}(\Delta\psi)^2\Delta w \\
& + (C_{127}x_3^2+C_{128}h^2)w_{,\gamma\nu}w_{,\gamma\nu}\Delta w+(C_{129}x_3^2+C_{130}h^2)(\Delta w)^3-2\mu T_{\gamma\nu}w_{,\gamma\nu} \\
& + \Big(C_{131}\frac{x_3^2}{h^2}+C_{132}\Big)\ T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi+C_{133}T_{\gamma\nu}T_{\nu\omega}w_{,\omega\gamma} \\
& - \frac{\lambda\mu}{3\lambda+2\mu}\ \Big(x_3^2-\frac{1}{3}\ h^2\Big)\Delta\psi_{,\gamma\nu}w_{,\gamma\nu}+(C_{134}x_3^2+2\mu C_{24}h^2)w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma} \\
& + \Big[\frac{\mu(3\lambda+4\mu)}{3(\lambda+2\mu)}\ x_3^2 - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\ h^2\Big]\Delta w_{,\gamma\nu}T_{\gamma\nu} - \mu T_{\gamma\nu}w_{,\gamma\nu}w_{,\omega}w_{,\omega} \\
& + 4\mu w_{,\omega}T_{\gamma\nu}T_{\omega\gamma,\nu} - 2\mu\psi_{,\gamma\nu}\psi_{,\gamma\nu\omega}w_{,\omega} - \frac{4\lambda\mu}{3\lambda+2\mu}\ T_{\gamma\nu}\Delta\psi_{,\gamma}w_{,\nu} \\
& + \frac{2\lambda\mu}{3\lambda+2\mu}\ \Delta\psi\ \Delta\psi_{,\gamma}w_{,\gamma}\Big\} + \mathcal{O}\Big(\frac{\mu h^6}{D^6}\ \varepsilon_1\Big)
\end{aligned}
$$

where

$$
(4.65)\ \left\{
\begin{aligned}
C_{125} &= \mu C_{113} - \frac{2\lambda\mu}{\lambda+2\mu}\ C_{121} - \frac{2\mu B}{\lambda+2\mu} = -\mu \\
C_{126} &= \mu C_{114} - \frac{2\lambda\mu}{\lambda+2\mu}\ C_{122} + \frac{2\lambda\mu}{3\lambda+2\mu}\ C_{68} - \frac{8\mu^3A-2\lambda\mu(3\lambda+8\mu)B+2\lambda^2\mu C}{(\lambda+2\mu)(3\lambda+2\mu)^2} \\
&= \frac{\lambda\mu}{3\lambda+2\mu} \\
C_{127} &= \mu C_{115} - \frac{2\lambda\mu}{\lambda+2\mu}\ C_9 - \frac{2\mu B}{\lambda+2\mu} = -\frac{\lambda^2\mu}{4(\lambda+\mu)(\lambda+2\mu)} - \frac{2\mu B}{\lambda+2\mu} \\
C_{128} &= \mu C_{116} - \frac{2\lambda\mu}{\lambda+2\mu}\ C_{123} = \frac{\mu(5\lambda^2-4\lambda\mu-8\mu^2)}{12(\lambda+\mu)(\lambda+2\mu)} + \frac{2\mu B}{\lambda+2\mu} \\
C_{129} &= \mu C_{117} - \frac{2\lambda\mu}{\lambda+2\mu}\ C_{10} - \frac{8\mu^3A+6\lambda^2\mu B-\lambda^3C}{3(\lambda+2\mu)^3} \\
&= \frac{\lambda^2\mu(5\lambda+\ 6\mu)}{4(\lambda+\mu)(\lambda+2\mu)^2} - \frac{8\mu^3A+6\lambda^2\mu B-\lambda^3C}{3(\lambda+2\mu)^3} \\
C_{130} &= \mu C_{118} - \frac{2\lambda\mu}{\lambda+2\mu}\ C_{124} \\
&= -\frac{\lambda\mu(17\lambda^2+26\lambda\mu+8\mu^2)}{12(\lambda+\mu)(\lambda+2\mu)^2} + \frac{8\mu^3A+6\lambda^2\mu B-\lambda^3C}{3(\lambda+2\mu)^3}
\end{aligned}
\right.
$$

$$\left|\begin{array}{l} C_{131} = \mu C_{119} + \frac{2\lambda\mu}{3\lambda+2\mu} C_{67} = 0 \\ \\ C_{132} = \mu C_{120} + \frac{2\lambda\mu}{3\lambda+2\mu} C_{66} + \frac{2\lambda C - 4\mu B}{3\lambda+2\mu} = - \frac{2\lambda\mu}{3\lambda+2\mu} \\ \\ C_{133} = 2\mu C_{22} - C = 4\mu \\ \\ C_{134} = -\mu - \frac{C}{3} \end{array}\right.$$

From (4.58) and (4.64)

$$\begin{aligned} (4.66)\quad p_{331} = x_3 \Big\{ & 2\mu \big(T_{\gamma\nu} w_{,\gamma\nu} + \tfrac{1}{2} \psi_{,\gamma\nu} \psi_{,\gamma\nu} \Delta w - 2T_{\gamma\nu} T_{\nu\omega} w_{,\omega\gamma} \\ & + \tfrac{1}{2} T_{\gamma\nu} w_{,\gamma\nu} w_{,\omega} w_{,\omega} - 2T_{\gamma\nu} T_{\omega\gamma,\nu} w_{,\omega} + \psi_{,\gamma\nu} \psi_{,\gamma\nu\omega} w_{,\omega} \big) \\ & + \frac{\lambda\mu}{3\lambda+2\mu} \Big[-(\Delta\psi)^2 \Delta w + 2 \frac{x_3^2}{h^2} T_{\gamma\nu} w_{,\gamma\nu} \Delta\psi + (x_3^2 - \tfrac{1}{3} h^2) \Delta\psi_{,\gamma\nu} w_{,\gamma\nu} \\ & + 4T_{\gamma\nu} \Delta\psi_{,\gamma} w_{,\nu} - 2\Delta\psi\, \Delta\psi_{,\gamma} w_{,\gamma} \Big] \\ & + (C_{135} x_3^2 - C_{136} h^2) w_{,\gamma\nu} w_{,\gamma\nu} \Delta w + (C_{137} x_3^2 - C_{138} h^2)(\Delta w)^3 \\ & + \big[(\mu + \tfrac{C}{3}) x_3^2 - \tfrac{1}{3} (5\mu + C) h^2 \big] w_{,\gamma\nu} w_{,\nu\omega} w_{,\omega\gamma} \\ & - \Big[\frac{\mu(3\lambda+4\mu)}{3(\lambda+2\mu)} x_3^2 - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} h^2 \Big] \Delta w_{,\gamma\nu} T_{\gamma\nu} \Big\} + \mathcal{O}\Big(\frac{\mu h^6}{D^6} \varepsilon_1\Big) \end{aligned}$$

where

$$(4.67)\left\{\begin{array}{l} C_{135} = \frac{2\lambda\mu}{\lambda+2\mu} - C_{127} = \frac{\lambda\mu(9\lambda+8\mu)}{4(\lambda+\mu)(\lambda+2\mu)} + \frac{2\mu B}{\lambda+2\mu} \\ \\ C_{136} = \frac{2\lambda\mu}{\lambda+2\mu} + C_{128} = \frac{\mu(29\lambda^2+20\lambda\mu-8\mu^2)}{12(\lambda+\mu)(\lambda+2\mu)} + \frac{2\mu B}{\lambda+2\mu} \\ \\ C_{137} = \frac{2\lambda^2\mu}{(\lambda+2\mu)^2} - C_{129} = \frac{\lambda^2\mu(3\lambda+2\mu)}{4(\lambda+\mu)(\lambda+2\mu)^2} + \frac{8\mu^3 A + 6\lambda^2\mu B - \lambda^3 C}{3(\lambda+2\mu)^3} \\ C_{138} = \frac{2\lambda^2\mu}{(\lambda+2\mu)^2} + C_{130} = \frac{\lambda\mu(7\lambda^2-2\lambda\mu-8\mu^2)}{12(\lambda+\mu)(\lambda+2\mu)^2} + \frac{8\mu^3 A + 6\lambda^2\mu B - \lambda^3 C}{3(\lambda+2\mu)^3} \end{array}\right.$$

Since $(t_{331}+p_{331})_{,3} = -(t_{3\alpha 1}+p_{3\alpha 1})_{,\alpha}$, then from (4.56) and (4.66) we can obtain $t_{331,3}$ and integrate from -h to x_3 to obtain

$$
\begin{aligned}
(4.68)\quad -t_{331} = {} & \frac{1}{2}\left(\frac{1}{3}x_3^3 - x_3h^2\right)[C_{101}w_{,\gamma}w_{,\gamma\nu}\Delta w_{,\nu} + C_{103}w_{,\gamma}\Delta w_{,\gamma}\Delta w \\
& + C_{105}w_{,\gamma}w_{,\nu\omega}w_{,\gamma\nu\omega} + C_{106}\Delta^2 w + C_{107}\Delta\psi_{,\gamma}\Delta w_{,\gamma}] \\
& + \frac{1}{2}\left[-\frac{1}{5}\mu\frac{x_3^5}{h^2} + \frac{1}{2}(\mu - C_{108})x_3^3 + C_{108}x_3h^2\right]\Delta T_{\gamma\nu}w_{,\gamma\nu} \\
& + \frac{1}{2}\left[-\frac{2}{5}\mu\frac{x_3^5}{h^2} + \frac{1}{3}(2\mu + C_{109})x_3^3 - C_{109}x_3h^2\right]T_{\gamma\nu,\omega}w_{,\gamma\nu\omega} \\
& + \left(\frac{1}{3}C_{139}x_3^3 - C_{140}x_3h^2\right)w_{,\gamma\nu}w_{,\gamma\nu}\Delta w + \left(\frac{1}{3}C_{141}x_3^3 + C_{142}x_3h^2\right)\Delta\psi_{,\gamma\nu}w_{,\gamma\nu} \\
& + \frac{1}{2}\mu\left(\frac{x_3^3}{h^2} - x_3\right)T_{\gamma\nu}w_{,\gamma\nu}w_{,\omega}w_{,\omega} \\
& + \left(\frac{1}{3}C_{143}x_3^3 - C_{144}x_3h^2 - \frac{\mu}{10}\frac{x_3^5}{h^2}\right)T_{\gamma\nu}\Delta w_{,\gamma\nu} \\
& + \left(\frac{1}{3}C_{145}x_3^3 - C_{146}x_3h^2\right)(\Delta w)^3 + \left(\frac{1}{3}C_{147}\frac{x_3^3}{h^2} - \frac{1}{2}C_{112}x_3\right)T_{\gamma\nu}w_{,\gamma\nu}\Delta\psi \\
& + \left[\frac{1}{3}(4\mu + C)x_3^3 - \frac{1}{3}(8\mu + C)x_3h^2\right]w_{,\gamma\nu}w_{,\nu\omega}w_{,\omega\gamma} \\
& + 2\mu x_3\left(T_{\gamma\nu}w_{,\gamma\nu} + \frac{1}{2}\psi_{,\gamma\nu}\psi_{,\gamma\nu}\Delta w - 2T_{\gamma\nu}T_{\nu\omega}w_{,\omega\gamma}\right. \\
& \left. - 2T_{\gamma\nu}T_{\omega\gamma,\nu}w_{,\omega} + \psi_{,\gamma\nu}\psi_{,\gamma\nu\omega}w_{,\omega}\right) \\
& + \frac{\lambda\mu}{3\lambda + 2\mu}x_3[4T_{\gamma\nu}\Delta\psi_{,\gamma}w_{,\nu} - (\Delta\psi)^2\Delta w - 2\Delta\psi\,\Delta\psi_{,\gamma}w_{,\gamma}] + \left(\frac{\mu h^6}{D^6}\varepsilon_1\right)
\end{aligned}
$$

where

$$
\left\{
\begin{aligned}
C_{139} &= 3C_{135} + \frac{1}{2} C_{102} = \frac{\lambda\mu(15\lambda+16\mu)}{2(\lambda+\mu)(\lambda+2\mu)} \\
&\quad + \frac{8\mu^3 A + 2\mu(13\lambda^2+32\lambda\mu+24\mu^2)B - \lambda^3 C}{4(\lambda+\mu)(\lambda+2\mu)^2} \\
C_{140} &= C_{136} + \frac{1}{2} C_{102} = \frac{\mu(19\lambda^2+22\lambda\mu-4\mu^2)}{6(\lambda+\mu)(\lambda+2\mu)} \\
&\quad + \frac{8\mu^3 A + 2\mu(5\lambda^2+8\lambda\mu+8\mu^2)B - \lambda^3 C}{4(\lambda+\mu)(\lambda+2\mu)^2} \\
C_{141} &= \frac{3\lambda\mu}{3\lambda+2\mu} + \frac{1}{2} C_{104} = \frac{2\lambda\mu}{\lambda+2\mu} + \frac{2\mu B - \lambda C}{3\lambda+2\mu} \\
C_{142} &= -\frac{\lambda\mu}{3(3\lambda+2\mu)} - \frac{1}{2} C_{104} = -\frac{2\lambda\mu(5\lambda-2\mu)}{3(\lambda+2\mu)(3\lambda+2\mu)} + \frac{\lambda C - 2\mu B}{3\lambda+2\mu} \\
C_{143} &= \frac{1}{2} C_{110} + \frac{\mu}{2} - \frac{\mu(3\lambda+4\mu)}{\lambda+2\mu} = \frac{15\lambda\mu+16\mu^2+8\mu B}{2(\lambda+2\mu)} + C \\
C_{144} &= \frac{1}{2} C_{110} - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} = \frac{6\lambda\mu+7\mu^2+4\mu B}{\lambda+2\mu} + C \\
C_{145} &= \frac{1}{2} C_{111} + 3C_{137} = \frac{\lambda\mu(5\lambda^2+2\lambda\mu-4\mu^2)}{2(\lambda+\mu)(\lambda+2\mu)^2} \\
&\quad + \frac{(8\mu^3 A - \lambda^3 C)(3\lambda+2\mu) + 2\lambda\mu(11\lambda^2+14\lambda\mu+8\mu^2)B}{4(\lambda+\mu)(\lambda+2\mu)^3} \\
C_{146} &= \frac{1}{2} C_{111} + C_{138} = \frac{\lambda\mu(5\lambda^2-4\lambda\mu-16\mu^2)}{6(\lambda+\mu)(\lambda+2\mu)^2} \\
&\quad + \frac{(8\mu^3 A - \lambda^3 C)(\lambda-2\mu) + 6\lambda\mu(3\lambda^2+6\lambda\mu+8\mu^2)B}{12(\lambda+\mu)(\lambda+2\mu)^3} \\
C_{147} &= \frac{1}{2} C_{112} + \frac{6\lambda\mu}{3\lambda+2\mu} \\
&= -\frac{3\lambda\mu}{3\lambda+2\mu} + \frac{12\mu^3 A + 6\mu(\lambda^2+2\mu^2)B - 3\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)}
\end{aligned}
\right.
\tag{4.69}
$$

If we subtract from (4.68) the equation obtained by setting $x_3 = h$ in (4.68) and recall that $t_{331} = 0$ for $x_3 = h$, then we obtain line 1 of (0.21) where

$$\left\{\begin{aligned}
C_{148} &= \frac{1}{2}(C_{139}-C_{140}) = \frac{\mu(13\lambda^2+13\lambda\mu+2\mu^2)}{6(\lambda+\mu)(\lambda+2\mu)} + \frac{2\mu B}{\lambda+2\mu} \\
C_{149} &= \frac{1}{2}(C_{141}+C_{142}) = \frac{4\lambda\mu}{3(3\lambda+2\mu)} \\
C_{150} &= \frac{1}{2}(C_{143}-C_{144}-\frac{3\mu}{10}) = \frac{\mu(3\lambda+\mu)}{5(\lambda+2\mu)} \\
C_{151} &= \frac{1}{2}(C_{145}-C_{146}) = \frac{\lambda\mu(5\lambda^2+5\lambda\mu+2\mu^2)}{6(\lambda+\mu)(\lambda+2\mu)^2} + \frac{8\mu^3A+6\lambda^2\mu B-\lambda^3C}{3(\lambda+2\mu)^3} \\
C_{152} &= \frac{1}{2}C_{147} - \frac{1}{4}C_{112} = \frac{3\lambda\mu}{3\lambda+2\mu}
\end{aligned}\right. \tag{4.70}$$

Also setting $x_3 = h$ in (4.68) and using the condition that $t_{331} = 0$ for $x_3 = h$, gives the second result of (0.22) where

$$\left\{\begin{aligned}
C_{153} &= \frac{1}{6\mu}(C_{147}-\frac{3}{2}C_{112}) = \frac{4\lambda}{3\lambda+2\mu} \\
&\quad - \frac{4\mu^3A+2\mu(\lambda^2+2\mu^2)B-\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{\mu(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} \\
C_{154} &= -\frac{1}{6\mu}C_{101} = -\frac{4(\lambda+\mu)}{3(\lambda+2\mu)} \\
C_{155} &= -\frac{1}{6\mu}C_{103} = \frac{\lambda}{3(\lambda+2\mu)} \\
C_{156} &= -\frac{1}{6\mu}C_{105} = -\frac{3\lambda+2\mu}{3(\lambda+2\mu)} \\
C_{157} &= -\frac{1}{6\mu}C_{107} = \frac{8\lambda(\lambda+\mu)}{3(\lambda+2\mu)(3\lambda+2\mu)} \\
&\quad - \frac{16\mu^3A+8\mu(\lambda^2+2\mu^2)B-4\lambda(\lambda^2+2\lambda\mu+2\mu^2)C}{3\mu(\lambda+2\mu)^2(3\lambda+2\mu)}
\end{aligned}\right. \tag{4.71}$$

$$\left|\begin{aligned}
C_{158} &= \frac{1}{6\mu}\left(\frac{1}{5}\mu + C_{108}\right) = \frac{-19\lambda+12\mu-20B}{30(\lambda+2\mu)}\\
C_{159} &= \frac{1}{6\mu}\left(\frac{2}{5}\mu - C_{109}\right) = -\frac{2(27\lambda+29\mu+10B)}{15(\lambda+2\mu)} - \frac{C}{3\mu}\\
C_{160} &= \frac{1}{6\mu}\left(C_{139}-3C_{140}\right) = -\frac{2\lambda-\mu}{6(\lambda+\mu)} - \frac{8\mu^3A+2\lambda\mu(\lambda-4\mu)B-\lambda^3C}{12\mu(\lambda+\mu)(\lambda+2\mu)^2}\\
C_{161} &= \frac{1}{6\mu}\left(C_{141}+3C_{142}\right) = -\frac{2\lambda(\lambda-2\mu)}{3(\lambda+2\mu)(3\lambda+2\mu)} + \frac{\lambda C-2\mu B}{3\mu(3\lambda+2\mu)}\\
C_{162} &= \frac{1}{6\mu}\left(C_{143}-3C_{144}-\frac{3}{10}\right) = -\frac{27\lambda+34\mu+20B}{15(\lambda+2\mu)} - \frac{C}{3\mu}\\
C_{163} &= \frac{1}{6\mu}\left(C_{145}-3C_{146}\right) = \frac{\lambda\mu}{2(\lambda+\mu)(\lambda+2\mu)} + \frac{8\mu^3A+2\lambda\mu(\lambda-4\mu)B-\lambda^3C}{12\mu(\lambda+\mu)(\lambda+2\mu)^2}
\end{aligned}\right.$$

From the second result of (0.19)

$$\begin{aligned}
(4.72)\quad t_{\alpha\beta 2,\beta} &= 2C_{33}\psi_{,\gamma\nu}\psi_{,\gamma\nu\alpha} + 2C_{34}\Delta\psi\Delta\psi_{,\alpha} + C_{37}T_{\alpha\beta}\Delta\psi_{,\beta} + C_{38}T_{\alpha\gamma,\beta}T_{\gamma\beta}\\
&+ (C_{164}x_3^2+C_{165}h^2)w_{,\gamma\nu}w_{,\gamma\nu\alpha} + (C_{166}x_3^2+C_{167}h^2)\Delta w\Delta w_{,\alpha}\\
&+ (C_{168}x_3^2+C_{169}h^2)w_{,\alpha\gamma}\Delta w_{,\gamma} + \left(\frac{\mu h^4}{D^5}\bar{\varepsilon}_2\right)
\end{aligned}$$

where

$$(4.73)\left\{\begin{aligned}
C_{164} &= 2C_{12}+C_{16}+\frac{\lambda\mu}{2(\lambda+\mu)} = \frac{6\lambda\mu+2\mu^2+4\mu B}{\lambda+2\mu} + C\\
C_{165} &= 2C_{35}-2\mu C_{24}-\frac{\lambda\mu}{6(\lambda+\mu)} = -\frac{8\lambda\mu+6\mu^2+4\mu B}{3(\lambda+2\mu)} - \frac{C}{3}\\
C_{166} &= 2C_{13}+C_{15}-\frac{\lambda\mu}{2(\lambda+\mu)} = -\frac{2\lambda\mu}{\lambda+2\mu} + \frac{16\mu^3A+16\mu(\lambda^2+\lambda\mu+\mu^2)B-2\lambda^3C}{(\lambda+2\mu)^3}\\
C_{167} &= 2C_{36}-2\mu C_{23}+\frac{\lambda\mu}{6(\lambda+\mu)} = -\frac{16\mu^3A+16\mu(\lambda^2+\lambda\mu+\mu^2)B-2\lambda^3C}{3(\lambda+2\mu)^3}\\
C_{168} &= C_{15}+C_{16} = \frac{2\mu^2+4\mu B}{\lambda+2\mu} + C\\
C_{169} &= -2\mu(C_{23}+C_{24}) = -\frac{8\lambda\mu+10\mu^2+4\mu B}{3(\lambda+2\mu)} - \frac{C}{3}\,.
\end{aligned}\right.$$

From (4.72) and the third result of (4.30)

$$(4.74)\quad (t_{\alpha\beta 2}+p_{\alpha\beta 2})_{,\beta} = (h^2-3x_3^2)\{C_{165}w_{,\gamma\nu}w_{,\gamma\nu\alpha}+C_{167}\Delta w\Delta w_{,\alpha}+C_{169}w_{,\alpha\gamma}\Delta w_{,\gamma}\} + \mathcal{O}\left(\frac{\mu h^4}{D^5}\bar{\varepsilon}_2\right).$$

Since $(t_{\alpha 32}+p_{\alpha 32})_{,3} = -(t_{\alpha\beta 2}+p_{\alpha\beta 2})_{,\beta}$, we obtain

$$(4.75)\quad t_{\alpha 32}+p_{\alpha 32} = x_3(x_3^2-h^2)\{C_{165}w_{,\gamma\nu}w_{,\gamma\nu\alpha}+C_{167}\Delta w\Delta w_{,\alpha}+C_{169}w_{,\alpha\gamma}\Delta w_{,\gamma}\} + \mathcal{O}\left(\frac{\mu h^5}{D^5}\bar{\varepsilon}_2\right) .$$

From (1.6)

$$(4.76)\quad p_{\alpha 32} = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}x_3(x_3^2-h^2)w_{,\alpha\gamma}\Delta w_{,\gamma} + \mathcal{O}\left(\frac{\mu h^5}{D^5}\bar{\varepsilon}_2\right)$$

so that the second result of (0.20) follows where

$$(4.77)\qquad C_{170} = C_{169} + \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} = \frac{4\lambda\mu+2\mu^2-4\mu B}{3(\lambda+2\mu)} - \frac{C}{3} .$$

From (1.4)

$$(4.78)\quad t_{\alpha 32} = 2\mu e_{\alpha 32} + x_3(x_3^2-h^2)\left\{\frac{\lambda+\mu}{(\lambda+2\mu)^2}(\lambda C-4\mu B)\Delta w\Delta w_{,\alpha} - \frac{\lambda+\mu}{\lambda+2\mu}Cw_{,\alpha\gamma}\Delta w_{,\gamma}\right\} + \mathcal{O}\left(\frac{\mu h^5}{D^5}\bar{\varepsilon}_2\right).$$

From (4.78) and the second result of (0.20) we obtain the second result of (0.17) where

$$(4.79)\left\{\begin{aligned} C_{171} &= \frac{1}{2\mu}C_{167} + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)^2}(4\mu B-\lambda C) \\ &= -\frac{16\mu^3 A+4\mu(\lambda^2-5\lambda\mu-2\mu^2)B+\lambda(\lambda^2+9\lambda\mu+6\mu^2)C}{6\mu(\lambda+2\mu)^3} \\ C_{172} &= \frac{1}{2\mu}C_{170} + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}C = \frac{2\lambda+\mu-2B}{3(\lambda+2\mu)} + \frac{2\lambda+\mu}{6\mu(\lambda+2\mu)}C .\end{aligned}\right.$$

From the second result of (0.17) and (4.2) we solve for $\mu_{\alpha 2,3}$ and integrate from 0 to x_3 to obtain the second result of (0.14) where

$$
(4.80)\left\{
\begin{aligned}
C_{173} &= -C_{43} - \frac{\lambda^2}{2(3\lambda+2\mu)^2} = \frac{\lambda^2}{(3\lambda+2\mu)^2} + \frac{4\mu^3 A - 3\lambda\mu(3\lambda+4\mu)B + 3\lambda^2(\lambda+\mu)C}{\mu(3\lambda+2\mu)^3} \\
C_{174} &= \frac{1}{4\mu} C_{165} - \frac{1}{6} C_7 = -\frac{(9\lambda+8\mu)(3\lambda+4\mu)}{48(\lambda+\mu)(\lambda+2\mu)} - \frac{B}{6(\lambda+2\mu)} - \frac{C}{12\mu} \\
C_{175} &= \frac{-1}{2\mu} C_{165} - C_{41} = \frac{89\lambda^3 + 274\lambda^2\mu + 280\lambda\mu^2 + 96\mu^3}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} \\
&\quad + \frac{4(\lambda+\mu)B}{3(\lambda+2\mu)(3\lambda+2\mu)} + \frac{\lambda+\mu}{3\mu(3\lambda+2\mu)} C \\
C_{176} &= \frac{1}{2} C_{171} - \frac{1}{2} C_{44} - \frac{\lambda^2}{8(\lambda+2\mu)^2} = -\frac{\lambda(7\lambda^2+16\lambda\mu+8\mu^2)}{48(\lambda+\mu)(\lambda+2\mu)^2} \\
&\quad + \frac{-8\mu^3 A + 2\mu(-\lambda^2+6\lambda\mu+4\mu^2)B - \lambda(\lambda+6\mu)(\lambda+\mu)C}{12\mu(\lambda+2\mu)^3} \\
C_{177} &= -C_{171} - C_{42} = \frac{\lambda(23\lambda^2+46\lambda\mu+24\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} \\
&\quad + \frac{(\lambda+\mu)[32\mu^3 A - 4\mu(\lambda^2+16\lambda\mu+4\mu^2)B + \lambda(5\lambda^2+24\lambda\mu+12\mu^2)C]}{6\mu(\lambda+2\mu)^3(3\lambda+2\mu)} \\
C_{178} &= \frac{1}{2} C_{172} - \frac{1}{4} C_{45} + \frac{3\lambda+4\mu}{8(\lambda+2\mu)} = \frac{7\lambda+6\mu-4B}{12(\lambda+2\mu)} + \frac{2\lambda+\mu}{12\mu(\lambda+2\mu)} C \\
C_{179} &= -C_{172} - \frac{1}{2} C_{46} - \frac{\lambda+\mu}{\lambda+2\mu} = -\frac{2\lambda+\mu-2B}{3(\lambda+2\mu)} - \frac{2\lambda+\mu}{6\mu(\lambda+2\mu)} C
\end{aligned}
\right.
$$

From (4.75)

$$
\begin{aligned}
(4.81)\quad (t_{3\alpha 2}+p_{3\alpha 2})_{,\alpha} &= x_3(x_3^2-h^2)\{C_{165} w_{,\gamma\nu\omega} w_{,\gamma\nu\omega} + (C_{165}+C_{169}) w_{,\gamma\nu} \Delta w_{,\gamma\nu} \\
&\quad + (C_{167}+C_{169}) \Delta w_{,\gamma} \Delta w_{,\gamma} + \frac{3(\lambda+2\mu)}{2(\lambda+\mu)} C_{167} \frac{-1}{h^2} T_{\gamma\nu} w_{,\gamma\nu} \Delta w\} + \mathcal{O}\left(\frac{\mu h^5}{D^6} \bar{\varepsilon}_2\right).
\end{aligned}
$$

From (1.6), (1.3), and (1.2)

$$\left\{\begin{aligned} p_{332} &= 2e_{3k1}t_{k31} + 2e_{332}t_{332} - W_e + \mathcal{O}\left(\frac{\mu h^8}{D^8}\,\bar{\varepsilon}_2\right) \\ W_e &= \frac{\lambda}{2}\,(s_{1e}^2+s_{1o}^2) + \mu s_{2e} + \frac{A}{3}\,(s_{1e}^3+3s_{1e}s_{1o}^2) \\ &\quad + B(s_{1e}s_{2e}+s_{1o}s_{2o}) + \frac{C}{3}\,s_{3e} + \mathcal{O}\left(\frac{\mu h^6}{D^6}\,\bar{\varepsilon}_2\right) . \end{aligned}\right. \tag{4.82}$$

From (4.82) and (1.1)

$$\begin{aligned} p_{332,3} = x_3\Big\{ &-2\mu w_{,\gamma\nu}w_{,\gamma\nu} - \frac{2\lambda\mu}{\lambda+2\mu}\,(\Delta w)^2 \\ &- \Big[\frac{4\lambda\mu}{3\lambda+2\mu}\,(C_{10}-C_8) + \frac{4\lambda\mu}{\lambda+2\mu}\,(C_3-C_{68}) + \frac{\lambda^2\mu(13\lambda+14\mu)}{2(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} \\ &+ \frac{16\mu^3A+4\lambda\mu(\lambda-4\mu)B-2\lambda^3C}{(\lambda+2\mu)^2(3\lambda+2\mu)}\Big]\,\Delta\psi(\Delta w)^2 \\ &- \Big[\frac{4\lambda\mu}{3\lambda+2\mu}\,(C_9-C_7) - \frac{\lambda\mu(5\lambda+4\mu)}{2(\lambda+\mu)(3\lambda+2\mu)} + \frac{4\mu B-2\lambda C}{3\lambda+2\mu}\Big]\,\Delta\psi w_{,\gamma\nu}w_{,\gamma\nu} \\ &- \Big[4B+2\lambda(C_1-C_{66})-3\lambda+4\lambda(1+C_2-C_{67})\,\frac{x_3^2}{h^2}\Big]\,\frac{2\mu}{\lambda+2\mu}\,T_{\gamma\nu}w_{,\gamma\nu}\Delta w \\ &- 2(\mu+C)w_{,\gamma\nu}w_{,\nu\omega}T_{\omega\gamma} - \frac{8\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\,w_{,\gamma}\Delta\psi_{,\gamma}\Delta w \\ &- \frac{2\lambda\mu}{\lambda+2\mu}\,w_{,\gamma}w_{,\gamma}(\Delta w)^2 + \Big[\frac{4\mu(3\lambda+4\mu)}{3(\lambda+2\mu)}\,x_3^2 - \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\,h^2\Big]w_{,\gamma\nu}\Delta w_{,\gamma\nu} \\ &- 2\mu w_{,\gamma}w_{,\gamma}w_{,\nu\omega}w_{,\nu\omega} - 4\mu w_{,\gamma}w_{,\nu\omega}(T_{\nu\omega,\gamma} - 2T_{\gamma\nu,\omega}) \\ &- \frac{8\lambda\mu}{3\lambda+2\mu}\,\Delta\psi_{,\gamma}w_{,\nu}w_{,\gamma\nu} - \frac{2\lambda\mu}{3\lambda+2\mu}\,T_{\gamma\nu}\Delta\psi_{,\gamma\nu} \\ &+ \frac{8\mu(\lambda+\mu)^2}{(\lambda+2\mu)^2}\,(x_3^2-h^2)\Delta w_{,\gamma}\Delta w_{,\gamma}\Big\} + \mathcal{O}\left(\frac{\mu h^5}{D^6}\,\bar{\varepsilon}_2\right). \end{aligned} \tag{4.83}$$

Since $(t_{332}+p_{332})_{,3} = -(t_{3\alpha2}+p_{3\alpha2})_{,\alpha}$, then (4.81) and (4.83) can be used to obtain

$$
\begin{aligned}
(4.84)\quad t_{332,3} = x_3\Big\{ & 2\mu w_{,\gamma\nu} w_{,\gamma\nu} + \frac{2\lambda\mu}{\lambda+2\mu}(\Delta w)^2 + 2C_{182}\Delta\psi(\Delta w)^2 \\
& + 2C_{183}\Delta\psi w_{,\gamma\nu} w_{,\gamma\nu} + \Big(2C_{184} + 4C_{185}\frac{x_3^2}{h^2}\Big) T_{\gamma\nu} w_{,\gamma\nu}\Delta w \\
& + 2(\mu+C) w_{,\gamma\nu} w_{,\nu\omega} T_{\omega\gamma} + \frac{8\lambda^2\mu}{(\lambda+2\mu)(3\lambda+2\mu)} w_{,\gamma}\Delta\psi_{,\gamma}\Delta w \\
& + \frac{2\lambda\mu}{\lambda+2\mu} w_{,\gamma} w_{,\gamma}(\Delta w)^2 + (4C_{186}x_3^2 + 2C_{187}h^2) w_{,\gamma\nu}\Delta w_{,\gamma\nu} \\
& + 2\mu w_{,\gamma} w_{,\gamma} w_{,\nu\omega} w_{,\nu\omega} + 4\mu w_{,\gamma} w_{,\nu\omega}(T_{\nu\omega,\gamma} - 2T_{\gamma\nu,\omega}) \\
& + \frac{8\lambda\mu}{3\lambda+2\mu}\Delta\psi_{,\gamma} w_{,\nu} w_{,\gamma\nu} + \frac{2\lambda\mu}{3\lambda+2\mu} T_{\gamma\nu}\Delta\psi_{,\gamma\nu} \\
& + 4C_{188}(x_3^2-h^2)\Delta w_{,\gamma}\Delta w_{,\gamma} - C_{165}(x_3^2-h^2) w_{,\gamma\nu\omega} w_{,\gamma\nu\omega}\Big\} + \mathcal{O}\Big(\frac{\mu h^5}{D^6}\bar{\varepsilon}_2\Big)
\end{aligned}
$$

where

$$
(4.85)\quad \left\{
\begin{aligned}
C_{182} &= \frac{2\lambda\mu}{3\lambda+2\mu}(C_{10}-C_8) + \frac{2\lambda\mu}{\lambda+2\mu}(C_3-C_{68}) + \frac{\lambda^2\mu(13\lambda+14\mu)}{4(\lambda+\mu)(\lambda+2\mu)(3\lambda+2\mu)} \\
&\quad + \frac{8\mu^3A + 2\lambda\mu(\lambda-4\mu)B - \lambda^3C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\
&= \frac{8\mu^3A + 2\lambda\mu(\lambda-4\mu)B - \lambda^3C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\
C_{183} &= \frac{2\lambda\mu}{3\lambda+2\mu}(C_9-C_7) - \frac{\lambda\mu(5\lambda+4\mu)}{4(\lambda+\mu)(3\lambda+2\mu)} + \frac{2\mu B-\lambda C}{3\lambda+2\mu} = \frac{2\mu B-\lambda C}{3\lambda+2\mu} \\
C_{184} &= [4B + 2\lambda(C_1-C_{66}) - 3\lambda]\frac{\mu}{\lambda+2\mu} + \frac{3(\lambda+2\mu)}{4(\lambda+\mu)}C_{167} \\
&= \frac{3\lambda\mu}{\lambda+2\mu} - \frac{8\mu^3A - 8\mu^2(2\lambda+\mu)B - \lambda^3C}{2(\lambda+\mu)(\lambda+2\mu)^2} \\
C_{185} &= (1+C_2-C_{67})\frac{2\lambda\mu}{\lambda+2\mu} - \frac{3(\lambda+2\mu)}{8(\lambda+\mu)}C_{167} \\
&= \frac{\lambda^2\mu}{2(\lambda+\mu)(\lambda+2\mu)} + \frac{8\mu^3A + 8\mu(\lambda^2+\lambda\mu+\mu^2)B - \lambda^3C}{4(\lambda+\mu)(\lambda+2\mu)^2}
\end{aligned}
\right.
$$

$$\left\{\begin{aligned} C_{186} &= \frac{-\mu(3\lambda+4\mu)}{3(\lambda+2\mu)} - \frac{1}{4}(C_{165}+C_{169}) = \frac{\lambda\mu+2\mu B}{3(\lambda+2\mu)} + \frac{C}{6} \\ C_{187} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} + \frac{1}{2}(C_{165}+C_{169}) = = \frac{4\mu(\lambda+\mu-B)}{3(\lambda+2\mu)} - \frac{C}{3} \\ C_{188} &= -\frac{2\mu(\lambda+\mu)^2}{(\lambda+2\mu)^2} - \frac{1}{4}(C_{167}+C_{169}) \\ &= -\frac{\mu(8\lambda^2+11\lambda\mu+2\mu^2)}{6(\lambda+2\mu)^2} \\ &\quad + \frac{16\mu^3A+4\mu(5\lambda^2+8\lambda\mu+8\mu^2)B-(\lambda^3-6\lambda^2\mu-12\lambda\mu^2-8\mu^3)C}{12(\lambda+2\mu)^3} \end{aligned}\right.$$

Integrating (4.84) from -h to x_3 gives the second result of (0.21) where

$$\left\{\begin{aligned} C_{189} &= C_{184} + C_{185} = \frac{\lambda\mu(7\lambda+6\mu)}{2(\lambda+\mu)(\lambda+2\mu)} - \frac{8\mu^3A-8\mu(\lambda^2+5\lambda\mu+3\mu^2)B-\lambda^3C}{4(\lambda+\mu)(\lambda+2\mu)^2} \\ C_{190} &= C_{186} + C_{187} = \frac{\mu(5\lambda+4\mu-2B)}{3(\lambda+2\mu)} - \frac{C}{6} \,. \end{aligned}\right. \tag{4.86}$$

This completes the proof of (0.13) through (0.22). The proof of (0.13') through (0.22') follows trivially using the definitions of v_α , w, $T_{\alpha\beta}$, ψ and (0.5) through (0.7)

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N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York, N. Y. 10012